Large Independent Sets in General Random Intersection Graphs ⋆

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Abstract

We investigate the existence and efficient algorithmic construction of close to optimal independent sets in random models of intersection graphs. In particular, (a) we propose *a new model* for random intersection graphs ($G_{n,m,\vec{p}}$) which includes the model of [10] (the "uniform" random intersection graphs model) as an important special case. We also define an interesting variation of the model of random intersection graphs, similar in spirit to random regular graphs. (b) For this model we derive *exact formulae* for the mean and variance of the number of independent sets of size k (for any k) in the graph. (c) We then propose and analyse *three algorithms* for the efficient construction of large independent sets in this model. The first two are variations of the greedy technique while the third is a totally new algorithm. Our algorithms are analysed for the special case of uniform random intersection graphs.

Our analyses show that these algorithms succeed in finding *close to optimal* independent sets for an interesting range of graph parameters.

Key words: random intersection graph, randomized algorithm, independent set, wireless sensor networks

1 Introduction

Random graphs, introduced by P. Erdös and A. Rényi, still continue to attract a huge amount of research and interest in the communities of Theoretical

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There exist various models of random graphs. The most famous is the $G_{n,p}$ random graph, a sample space whose points are graphs produced by randomly sampling the edges of a graph on n vertices independently, with the same probability p. Other models have also been quite a lot investigated: $G_{n,r}$ (the "random regular graphs", produced by randomly and equiprobably sampling a graph from all regular graphs of n vertices and vertex degree r) and $G_{n,M}$ (produced by randomly and equiprobably selecting an element of the class of graphs on n vertices having M edges). For an excellent survey of these models, see [1,3].

In this work we investigate, both combinatorially and algorithmically, a *new model* of random graphs. We nontrivially extend the $G_{n,m,p}$ model ("random intersection graphs") introduced by M. Karoński, E.R. Sheinerman and K.B. Singer-Cohen [10] and K.B. Singer-Cohen [17]. Also, Godehardt and Jaworski [9] considered similar models. In the $G_{n,m,p}$ model, to each of the n vertices of the graph, a random subset of a universal set of m elements is assigned, by independently choosing elements with the same probability p. Two vertices u, v are then adjacent in the $G_{n,m,p}$ graph if and only if their assigned sets of elements have at least one element in common. We extend this model (which we call hereafter "uniform", because of the same probability of selecting elements) by proposing *two new models* which we define below.

Definition 1 (General random intersection graph) *Let us consider a universe $M = \{1, 2, \ldots, m\}$ of elements and a set of vertices $V = \{v_1, v_2, \ldots, v_n\}$. If we assign independently to each vertex v_j, $j = 1, 2, \ldots, n$, a subset S_{v_j} of M by choosing each element $i \in M$ independently with probability p_i, $i = 1, 2, \ldots, m$, and put an edge between two vertices v_{j_1}, v_{j_2} if and only if $S_{v_{j_1}} \cap S_{v_{j_2}} \neq \emptyset$, then the resulting graph is an instance of the general random intersection graph $G_{n,m,\vec{p}}$, where $\vec{p} = [p_1, p_2, \ldots, p_m]$.*

Definition 2 (Regular random intersection graph) *Let us consider a universe $M = \{1, 2, \ldots, m\}$ of elements and a set of vertices $V = \{v_1, v_2, \ldots, v_n\}$. If we assign independently to each vertex v_j, $j = 1, 2, \ldots, n$, a subset S_{v_j} consisting of λ different elements of M, randomly and uniformly chosen, and draw an edge between two vertices v_{j_1}, v_{j_2} if and only if $S_{v_{j_1}} \cap S_{v_{j_2}} \neq \emptyset$, then the resulting graph is an instance of the regular random intersection graph $G_{n,m,\lambda}$.*

The latter model may abstract λ-SAT random formulae, in the sense that vertices in our model here correspond to clauses and labels to literals. We note the following:

Note 1: When $p_1 = p_2 = \cdots = p_m = p$ the general random intersection graph $G_{n,m,\vec{p}}$ reduces to the $G_{n,m,p}$ as in [10] and we call it the *uniform* random intersection graph.

Note 2: When in the uniform case $mp \geq \alpha \log n$ for some constant $\alpha > 1$ then the model $G_{n,m,p}$ and the model $G_{n,m,\lambda}$ for $\lambda \in (1 \pm \epsilon)mp$, $\epsilon \in (0, 1)$, are similar in the sense that, by using concentration arguments proved by Chernoff bounds, we can show that with high probability, the number of labels hit by a vertex in each model is almost the same. This may be used to translate properties from one space to the other. The investigation of exact equivalence properties of the two models is an interesting problem.

Importance and Motivation. First of all, we note that (as proved in [11]) any graph is a random intersection graph. Thus, the $G_{n,m,p}$ model is very general. Furthermore, for some ranges of the parameters m, p ($m = n^{\alpha}, \alpha > 6$) the spaces $G_{n,m,p}$ and $G_{n,p}$ are equivalent (as proved by Fill, Sheinerman and Singer-Cohen [8], showing that in this range the total variation distance between the graph random variables has limit 0).

Second, random intersection graphs (and in particular our new, non- uniform model) may model real-life applications more accurately (compared to the $G_{n,p}$ case). In fact there are practical situations where each communication agent (e.g. a wireless node) gets access only to some ports (statistically) out of a possible set of communication ports. When another agent also selects a communication port, then a communication link is implicitly established and this gives rise to communication graphs that look like random intersection graphs. Even epidemiological phenomena (like spread of disease) tend to be more accurately captured by this "proximity-sensitive" random intersection graphs model. Other applications may include oblivious resource sharing in a distributed setting, interactions of mobile agents traversing the web etc.

Other Related Work. The question of how close $G_{n,m,p}$ and $G_{n,p}$ are for various values of m, p has been studied by Fill, Sheinerman and Singer-Cohen in [8]. Also, geometric proximity between randomly placed objects is nicely captured by the model of random geometric graphs (see e.g. [4,7,15]) and important variations (like random scaled sector graphs, [6]). Other extensions of random graph models (such as random regular graphs) and several important combinatorial properties (connectivity, expansion, existence of a giant connected component) are performed in [12,14].

Our Contribution.

(1) We first introduce *two new models*, as explained above: the $G_{n,m,\vec{p}}$ model and the $G_{n,m,\lambda}$ model. We feel that our models are important, in the sense that $G_{n,m,\vec{p}}$ is a very general model and $G_{n,m,\lambda}$ is very focused (so it is particularly precise in abstracting several phenomena).
(2) Under these models we study the well known and fundamental problem of *finding a maximum independent set of vertices.* In particular, in the most general $G_{n,m,\vec{p}}$ model we estimate *exactly* the mean and the variance

of the number of independent sets of size k. To get exact formulas for the variance, we introduce and use a *"vertex contraction technique"* to evaluate the covariance of random indicator variables of non-disjoint sets of vertices. This technique, we believe, has its own combinatorial interest and may be used in investigating other combinatorial problems as well.

(3) Finally, we provide and analyse *three efficient algorithms* for finding large independent sets:
   - Algorithm I is the classic greedy algorithm (for example see [2]) for maximum independent set approximation.
   - Algorithm II is a variation of the above where a random new vertex is tried each time instead of that of current minimum degree.
   - Algorithm III is a totally new algorithm (that we propose) pertinent to the model $G_{n,m,\vec{p}}$.

   For clarity, all our algorithms are analysed for the uniform random intersection graphs model.

   Our algorithms are analysed for the interesting case where $mp \geq \alpha \log n$, (for some constant $\alpha > 1$), in which no isolated vertices exist in $G_{n,m,p}$ and also the results translate to $G_{n,m,\lambda}$ (see Note 2).

   To our knowledge, this is the first time that algorithms for random intersection graphs are proposed and analysed. Our analyses show that in many interesting ranges of p, m, the sizes of the independent sets obtained by the algorithms are quite large.

2 The Size of Independent Sets - Exact Formulae

In this section we compute the mean and variance of the number of independent sets of size k. Towards this, we provide several interesting techniques some of which we later use in the algorithms.

The following theorem gives an exact formula for the mean number of independent sets of size k in a general random intersection graph. In order to prove it, we view the graph from the point of view of its labels.

Theorem 3 *Let $X^{(k)}$ denote the number of independent sets of size k in a random intersection graph $G(n, m, \vec{p})$, where $\vec{p} = [p_1, p_2, \ldots, p_m]$. Then*

$$E\left[X^{(k)}\right] = \binom{n}{k} \prod_{i=1}^{m} \left((1-p_i)^k + kp_i(1-p_i)^{k-1}\right).$$

PROOF. Let V' be any set of k vertices and let

$$X_{V'} = \begin{cases} 1 \text{ if } V' \text{ is an independent set} \\ 0 \text{ otherwise.} \end{cases}$$

Clearly,

$$X^{(k)} = \sum_{V', |V'|=k} X_{V'}$$

and by the linearity of expectation

$$E\left[X^{(k)}\right] = \binom{n}{k} E\left[X_{V'}\right] = \binom{n}{k} P\{V' \text{ is an independent set}\}.$$

In order to determine $E\left[X_{V'}\right]$, let us look at the $G(n, m, \vec{p})$ graph from the point of view of the elements of $M = \{1, 2, \ldots, m\}$. The set V' will be an independent set if and only if every element of M is chosen by at most one of the k vertices in V'. Since the elements of M are chosen independently, it follows that

$$E\left[X_{V'}\right] = \prod_{i=1}^{m} P\{\text{element } i \text{ is chosen at most once by the vertices in } V'\}.$$

However, it is obvious that when a specific element i of M is chosen at most once by the vertices in V', then it is either chosen by exactly one vertex in V' or it is not chosen by any of them. Hence,

$$E\left[X_{V'}\right] = \prod_{i=1}^{m} (P\{\text{no vertex in } V' \text{ chooses i}\} + P\{\text{exactly one vertex chooses i}\}).$$

The probability that no vertex in V' chooses element i is exactly $(1 - p_i)^k$, which follows from the observation that each of the k vertices of V' chooses i with probability p_i and independently of the choices of other vertices.

Furthermore, the probability that exactly one vertex in V' chooses element i is exactly $kp_i(1 - p_i)^{k-1}$, since there are k different vertices in V' and the probability that only one particular vertex chooses i is $p_i(1 - p_i)^{k-1}$. We have therefore proven that

$$E\left[X^{(k)}\right] = \binom{n}{k} \prod_{i=1}^{m} \left((1 - p_i)^k + kp_i(1 - p_i)^{k-1}\right).$$

$\square$

We now prove a theorem that gives an exact formula for the variance of the number of independent sets of size k in a general random intersection graph. The proof uses a somewhat algorithmic technique that merges several independent vertices into a single supervertex.

Theorem 4 *Let $X^{(k)}$ denote the number of independent sets of size k in a random intersection graph $G(n, m, \vec{p})$, where $\vec{p} = [p_1, p_2, \dots, p_m]$. Then*

$$Var\left(X^{(k)}\right) = \sum_{s=1}^{k} \binom{n}{2k-s} \binom{2k-s}{s} \left(\gamma(k,s) \frac{E\left[X^{(k)}\right]}{\binom{n}{k}} - \frac{E^2\left[X^{(k)}\right]}{\binom{n}{k}^2} \right)$$

where $E\left[X^{(k)}\right]$ is the mean number of independent sets of size k and

$$\gamma(k,s) = \prod_{i=1}^{m} \left((1-p_i)^{k-s} + (k-s)p_i(1-p_i)^{k-s-1} \left(1 - \frac{sp_i}{1+(k-1)p_i} \right) \right).$$

PROOF. Let V' be any set of k vertices and let

$$X_{V'} = \begin{cases} 1 \text{ if } V' \text{ is an independent set} \\ 0 \text{ otherwise.} \end{cases}$$

Clearly, $X^{(k)} = \sum_{V',|V'|=k} X_{V'}$ and for V_1', V_2' any sets of k vertices,

$$\begin{aligned} Var\left(X^{(k)}\right) &= \sum_{V_1',V_2',|V_1'|=|V_2'|=k} Cov(X_{V_1'}, X_{V_2'}) \\ &= \sum_{s=1}^{k} \sum_{V_1',V_2',|V_1'|=|V_2'|=k,|V_1'\cap V_2'|=s} P\{X_{V_1'}X_{V_2'}=1\} - \frac{E^2\left[X^{(k)}\right]}{\binom{n}{k}^2}. \quad (1) \end{aligned}$$

Since

$$P\{X_{V_1'}X_{V_2'}=1\} = P\{X_{V_1'}=1|X_{V_2'}=1\} \frac{E\left[X^{(k)}\right]}{\binom{n}{k}} \quad (2)$$

the problem of computing the variance of $X^{(k)}$ is reduced to computing the conditional probability $P\{X_{V_1'}=1|X_{V_2'}=1\}$, i.e. the probability that V_1' is an independent set given that V_2' is an independent set, where V_1', V_2' are any two sets of k vertices that have s vertices in common. In order to compute $P\{X_{V_1'}=1|X_{V_2'}=1\}$, we will try to merge several vertices into one supervertex and study its probabilistic behaviour.

Towards this goal, let us fix an element i of $M=\{1,2,\ldots,m\}$ and let us consider two (super)vertices v_1, v_2 of the $G(n,m,\vec{p})$ graph that choose element i independently with probability $p_i^{(1)}$ and $p_i^{(2)}$ respectively. Let also S_{v_1}, S_{v_2} denote the sets of elements of M assigned to v_1 and v_2 respectively. Then,

$$\begin{aligned}P\{i\in S_{v_1}|\nexists(v_1,v_2)\} &= P\{i\in S_{v_1}, i\notin S_{v_2}|\nexists(v_1,v_2)\}\\ &= \frac{P\{i\in S_{v_1}, i\notin S_{v_2}, \nexists(v_1,v_2)\}}{P\{\nexists(v_1,v_2)\}} = \frac{p_i^{(1)}(1-p_i^{(2)})}{1-p_i^{(1)}p_i^{(2)}}\end{aligned} \tag{3}$$

where (v_1,v_2) is an edge between v_1 and v_2. From this we get:

- Conditional on the fact that (v_1,v_2) does not exist, the probabilistic behaviour of vertex v_1 is identical to that of a single vertex that chooses element i of M independently with probability $\frac{p_i^{(1)}(1-p_i^{(2)})}{1-p_i^{(1)}p_i^{(2)}}$.
- Conditional on the fact that (v_1,v_2) does not exist, the probabilistic behaviour of v_1 and v_2 considered as a unit is identical to that of a single vertex that chooses element i of M independently with probability

$$\begin{aligned}P\{i\in S_{v_1}\cup S_{v_2}|\nexists(v_1,v_2)\} &= P\{i\in S_{v_1}|\nexists(v_1,v_2)\}+P\{i\in S_{v_2}|\nexists(v_1,v_2)\}\\ &= \frac{p_i^{(1)}+p_i^{(2)}-2p_i^{(1)}p_i^{(2)}}{1-p_i^{(1)}p_i^{(2)}}\end{aligned} \tag{4}$$

  where i is a fixed element of M. The first of the above equations follows from the observation that if there is no edge between v_1 and v_2 then the sets S_{v_1} and S_{v_2} are disjoint, meaning that element i cannot belong to both of them. The second equation follows from symmetry.

Let us now consider merging one by one the vertices of the $G(n,m,\vec{p})$ graph into one supervertex. Let w_j denote a supervertex of j simple vertices that form an independent set. It is obvious that the probabilistic behaviour of w_j is irrelevant to how partial mergings are made. Moreover, if w_{j_1}, w_{j_2} are two supervertices representing two disjoint sets of simple vertices, we say that an edge (w_{j_1}, w_{j_2}) exists iff any edge connecting a simple vertex in w_{j_1} and a simple vertex in w_{j_2} exists. Thus, the event $\{\nexists(w_{j_1}, w_{j_2})\}$ is equivalent to the event $\{$the vertices in w_{j_1} together with those in w_{j_2} form an independent set$\}$.

Using equation (4) one can show that $P\{i\in S_{w_2}\}=\frac{2p_i}{1+p_i}$, $P\{i\in S_{w_3}\}=\frac{3p_i}{1+2p_i}$ and by induction

$$P\{i\in S_{w_j}\}=\frac{jp_i}{1+(j-1)p_i} \tag{5}$$

where i is a fixed element of M and S_{w_j} is the union of all the sets of elements of M assigned to each simple vertex in w_j. More formally,

$$S_{w_j} = \bigcup_{v \in w_j} S_v$$

where v is a simple vertex and S_v is the set of elements of M assigned to v. Because of the definition of w_j, the subsets S_v in the above union are disjoint.

Thus, let V_1' be any set of k (simple) vertices and let V_2' be an independent set of k vertices that has s vertices in common with V_1'. Since there is no edge between any vertices in V_2', we can treat the $k-s$ vertices of V_2' not belonging to V_1' and the s vertices belonging to both V_1' and V_2' as two seperate supervertices w_{k-s} and w_s respectively that do not communicate by an edge. Hence, by equations (3), (4) and (5), the probabilistic behaviour of w_s is identical to that of a single vertex w_s' that chooses the elements of M independently with probabilities $\{p_i^{(w_s')}, i = 1, \ldots, m\}$ respectively, where

$$p_i^{(w_s')} = \frac{p_i^{(w_s)}(1 - p_i^{(w_{k-s})})}{1 - p_i^{(w_s)} p_i^{(w_{k-s})}} = \frac{sp_i}{1 + (k-1)p_i}. \tag{6}$$

Let now V'' be a set of $k-s$ simple vertices and a vertex identical to w_s'. Then, for a fixed element i of M, each of the $k-s$ simple vertices chooses i independently with probability p_i, while the supervertex w_s' chooses i independently with probability $p_i^{(w_s')}$. Similarly to Theorem 3 we get

$$P\{X_{V_1'} = 1 | X_{V_2'} = 1\} = P\{V'' \text{ is an independent set}\} \stackrel{def}{=} \gamma(k, s).$$

Hence, by equations (1) and (2), we get the result.

□

Using the second moment method and the results of this section, one may prove thresholds for the existence (with high probability) of independent sets of size k.

3 Finding Large Independent Sets in $G_{n,m,p}$

We start from the classic greedy approach, i.e. starting from the empty set we introduce (into the independent set under construction) each time the *minimum degree* vertex in the current graph and then delete it and its neighbours from the graph (Algorithm I).

Algorithm I:
Input: An instance $G(V, e(G))$ of $G_{n,m,p}$.
Output: An independent set V' of G.

(1) set $V' := \emptyset$;
(2) set $U := V$;
(3) **while** $U \neq \emptyset$ **do**
(4) **begin**
(5) let $x :=$ vertex of *minimum* degree in the graph induced by U;
(6) set $V' := V' \cup \{x\}$;
(7) eliminate x and all its neighbours from U;
(8) **end**
(9) **output** V';

3.0.1 The Expected Size of the Independent Set constructed.

As can be seen in e.g. [2], if $r = |V'|$ eventually, and $\delta = \frac{|e(G)|}{n}$, i.e. δ is the *density* of G,

$$r(2\delta + 1) \geq n. \tag{7}$$

This holds for any input graph G. Taking expectations we get $E[r(2\delta + 1)] \geq n$, where the expectation is taken over all instances of the distribution $G_{n,m,p}$ (notice that both r, δ are random variables).

The property "$\exists$ independent set of size r" is monotone decreasing on the number of edges, while the property "the density of G is δ" is monotone increasing. A special case of the FKG inequality states that if A is a monotone increasing property and B is a monotone decreasing property, then $P(A \cap B) \leq P(A)P(B)$ (see also [1]). From this we get $E[r\delta] \leq E[r] E[\delta]$ or equivalently $E[r(2\delta + 1)] = 2E[r\delta] + E[r] \leq 2E[r] E[\delta] + E[r] = E[r](2E[\delta] + 1)$.

Using the fact that $E[r(2\delta + 1)] \geq n$, we conclude that

$$E[r] \geq \frac{n}{2E[\delta] + 1} = \frac{n}{2\frac{E(|e(G)|)}{n} + 1}. \tag{8}$$

In order to compute the mean number of edges $E(|e(G)|)$, let us define the indicator random variables

$$X_{u,v} = \begin{cases} 1 \text{ if there is an edge } (u, v) \text{ in } G \\ 0 \text{ otherwise.} \end{cases}$$

Clearly,

$$|e(G)| = \sum_{u,v \in V, u \neq v} X_{u,v}$$

and by the linearity of expectation

$$E(|e(G)|) = \binom{n}{2} E\left[X_{u,v}\right] = \binom{n}{2} P\{X_{u,v} = 1\}.$$

But

$$P\{X_{u,v} = 1\} = P\{\exists i \in M : i \in S_u \cap S_v\} = 1 - (1 - p^2)^m$$

where S_u, S_v are the sets of elements of $M = \{1, 2, \ldots, m\}$ assigned to u, v respectively.

Hence,

$$E(|e(G)|) = \binom{n}{2} \left(1 - (1 - p^2)^m\right).$$

Applying the above result to inequality (8), we conclude the following

Lemma 5 *The expected cardinality of the independent set constructed by Algorithm I is at least*

$$\frac{n^2}{2\binom{n}{2}\left(1 - (1 - p^2)^m\right) + n} = \frac{n^2}{2E(|e(G)|) + n}.$$

The next result is easily derived from Lemma 5.

Corollary 6 (Sparse $G_{n,m,p}$ theorem) *For p such that $E(|e(G)|) = \Theta(n)$, the expected size of the independent set provided by Algorithm I is $\Theta(n)$.*

For example, if $p = \frac{\alpha}{\sqrt{nm}}$, where $0 < \alpha < 1$, then $E\left[r\right] \geq \frac{n}{\alpha}$.

Remark: The above analysis carries out in an almost similar way to the general random intersection graphs model.

3.0.2 A Concentration Result for Sparse Graphs.

We are interested in intersection graphs $G_{n,m,p}$ for p satisfying

$$\frac{\omega(n)}{n\sqrt{m}} \leq p \leq \sqrt{\frac{2\log n - \omega(n)}{m}}$$

for the smallest possible function $\omega(n) \to \infty$, as $n \to \infty$. This is the range for nontrivial graphs (see [8]).

We consider the case $p < \sqrt{\frac{1}{8nm}}$ which is in the range of nontrivial graphs. In the sequel we assume that $p(n) = \frac{c(n)}{m}$ where $c(n) \to \infty$, as $n \to \infty$. For example, since $c(n) = mp$, if we take p in the range of nontrivial graphs, then

$$\frac{\sqrt{m}}{n}\omega(n) \leq c(n) \leq \sqrt{2m\log n - \omega(n)m}. \tag{9}$$

A choice of $c(n)$ satisfying this is $c(n) = \alpha \log n$, where $\alpha > 1$, since, from (9), $\omega(n)$ must be less than $2\log n$.

Notice that our assumption $p < \sqrt{\frac{1}{8nm}}$ implies $c(n) = mp < \sqrt{\frac{m}{8n}}$ which is satisfied by $c(n) = \alpha \log n$, i.e. for $m > 8\alpha^2 n \log^2 n$.

Consider a vertex v and let S_v be the set of elements assigned to it. Using Chernoff bounds (see e.g. [5]) and Boole's inequality, for $mp = \alpha \log n$ and $\epsilon \in (0, 1)$, we get

$$P\{\exists v : ||S_v| - mp| \geq \epsilon mp\} \leq \sum_{v \in V} P\{||S_v| - mp| \geq \epsilon mp\} \leq n^{-\frac{\alpha\epsilon^2}{2}+1}.$$

If we choose the parameter α so that $\frac{\alpha\epsilon^2}{2} - 1 > 2$, then all vertices have each a number of chosen elements "around" mp with probability at least $1 - \frac{1}{n^2}$.

Let us condition $G_{n,m,p}$ on this event. Because of symmetry, the elements chosen by each vertex are otherwise uniform in $\{1, 2, \ldots, m\}$.

Consider a variation of Algorithm I (Algorithm II) where we select greedily each time a *random* vertex to insert from the random graph.

Algorithm II:
Input: An instance $G(V, e(G))$ of $G_{n,m,p}$.
Output: An independent set V' of G.

set $V' := \emptyset$;
set $U := V$;
while $U \neq \emptyset$ **do**
begin
  let $u :=$ a random vertex of U;
  $U := U - \{u\}$;
  let $S(V') := \cup_{u \in V'} S_u$;
  if (u intersects with any vertex in V') **then** reject u
  else $V' := V' \cup \{u\}$;
end

The difference between Algorithm I and Algorithm II is that in the latter we do not use the (useful) heuristic urging us to choose at each iteration the vertex with the current minimum degree. We will denote the size of the independent sets constructed by Algorithm I and Algorithm II by r_1 and r_2 respectively.

We now concentrate on estimation of r_2 with high probability. Clearly, after i successful node insertions into V' the following are true:

- $|S(V')| \in (1 \pm \epsilon)imp = (1 \pm \epsilon)ic(n)$.
- The next tried node u is rejected with probability

$$P_{rej} = 1 - \left(1 - \frac{|S(V')|}{m}\right)^{|S_u|}$$

  since each element $l \in S_u$ belongs also in $S(V')$ with probability $\frac{|S(V')|}{m}$, which in turn follows from independence and uniformity.

Combining these two observations we conclude that the probability that a vertex u is rejected after i successful insertions is

$$P_{rej} \leq \frac{|S(V')||S_u|}{m}$$

which is at most $\frac{(1+\epsilon)^2 ic^2(n)}{m}$, for any $\epsilon \in (0, 1)$, provided that $\frac{ic^2(n)}{m} \to 0$, i.e. provided that $i = o\left(\frac{m}{c^2(n)}\right)$. (Note also that, by the Bernoulli inequality, we have

$$P_{acc} = 1 - P_{rej} = \left(1 - \frac{|S(V')|}{m}\right)^{|S_u|} \geq 1 - \frac{|S(V')||S_u|}{m}$$

and when $\frac{|S(V')||S_u|}{m} \to 0$, then $P_{acc} \to 1$.)

Since $i \le n$ and $1+\epsilon < 2$, for any $\epsilon \in (0,1)$, we have that $P_{rej} < \frac{4nc^2(n)}{m}$. Moreover, since $mp < \sqrt{\frac{m}{8n}}$ by assumption, we get $P_{rej} < \frac{1}{2}$. Thus, the number r_2 of nodes that are successfuly inserted into V' is at least the number of successes of the Bernoulli $\mathcal{B}(n, \frac{1}{2})$. From Chernoff bounds then, for any $\beta > 0$ we have $r_2 \ge (1-\beta)\frac{n}{2}$ with probability at least $1 - \exp\{-\frac{\beta^2}{2}\frac{n}{2}\}$.

We eventually have (set $\beta = \frac{1}{2}$), by combining events, the following

Theorem 7 *Consider a random intersection graph $G_{n,m,p}$ with $p < \sqrt{\frac{1}{8nm}}$ and $mp = \alpha \log n$, ($\alpha > 1$ a constant). Then Algorithm II constructs an independent set of size at least $\frac{n}{4}$ with probability at least $1 - \frac{1}{2n^2}$.*

Comment: Intuitively, r_1 stochastically dominates r_2 and so Theorem 7 may also apply to Algorithm I.

3.1 Algorithm III

Consider now the following algorithm that looks at the random intersection graph from the point of view of its labels.

Algorithm III:
Input: A random intersection graph $G_{n,m,p}$.
Output: An independent set of verices A_m.

(1) set $A_0 := V$; set $L := M$;
(2) **for** $i = 1$ **to** m **do**
(3) **begin**
(4) select a random label $l_i \in L$; set $L := L - \{l_i\}$;
(5) set $D_i := \{v \in A_{i-1} : l_i \in S_v\}$;
(6) **if** ($|D_i| \ge 1$) **then** select a random vertex $u \in D_i$ and set $D_i := D_i - \{u\}$;
(7) set $A_i := A_{i-1} - D_i$;
(8) **end**
(9) **output** A_m;

Theorem 8 (Correctness) *Algorithm III correctly finds an independent set of vertices.*

PROOF. In order to prove the correctness of Algorithm III let us consider any two vertices v_1 and v_2 that are connected via an edge, i.e. there is at least one element $i \in M$ that belongs to both S_{v_1} and S_{v_2}. It is easy to verify that at most one of these vertices can belong to A_m, since after the examination of element i of M, the algorithm will choose at most one of v_1 and v_2 (another possible scenario is that by the time the algorithm starts the examination of i one of the vertices v_1 and v_2 or both have been excluded from the independent set in previous steps).

□

Theorem 9 (Efficiency) *For the case $mp = \alpha \log n$ for some constant $\alpha > 1$ and $m \geq n$ with high probability we have for some constant $\beta > 0$:*

(1) If $np \to \infty$ then $|A_m| \geq (1-\beta)\frac{n}{\log n}$.
(2) If $np \to b$ where $b > 0$ is a constant then $|A_m| \geq (1-\beta)n(1-e^{-b})$.
(3) If $np \to 0$ then $|A_m| \geq (1-\beta)n$.

PROOF. Let us define the indicator random variables

$$X_v^{(i)} = \begin{cases} 1 \text{ if vertex } v \text{ of } A_{i-1} \text{ does not contain } l_i \\ 0 \text{ otherwise} \end{cases}$$

and

$$I_{D_i} = \begin{cases} 1 \text{ if } |D_i| \geq 1 \\ 0 \text{ otherwise.} \end{cases}$$

Clearly, $|A_i| = \sum_{v \in A_{i-1}} X_v^{(i)} + I_{D_i} \quad$, for $i = 1, 2, \ldots, m$.

Since the elements of M are chosen independently, the variables $X_v^{(i)}$ are independent of the set A_{i-1}. But Wald's equation states that if $Y_1, Y_2, \ldots$ are independent and identically distributed random variables having finite expectations, and if N is a stopping time for $Y_1, Y_2, \ldots$ such that $E[N] < \infty$, then

$$E\left[\sum_{j=1}^{N} Y_i\right] = E[N]E[Y].$$

Since the variables $X_v^{(i)}$ and the set A_{i-1} satisfy the conditions of Wald's equation (for the expectation of the sum of a random number of independent variables, see [16]), by the linearity of expectation we get

$$E(|A_i|) = E(|A_{i-1}|)(1-p) + P\{|D_i| \geq 1\} \quad , \text{for } i = 1, 2, \ldots, m.$$

Using the above equation we can prove inductively that

$$E(|A_m|) = n(1-p)^m + \sum_{i=1}^{m} (1-p)^{m-i} P\{|D_i| \geq 1\}. \tag{10}$$

(*Note:* By a similar proof one can verify that the term $n(1-p)^m$ is the mean number of isolated vertices in the graph. By choosing $mp \geq \alpha logn$ for some constant $\alpha > 1$ the mean number of isolated vertices tends to 0.)

Now let $L_i = \{v \in V : l_i \in S_v\}$, i.e. L_i is the set of vertices having l_i (before examining them for other elements of M). Then

$$P\{|D_i| \geq 1\} = 1 - P\{|D_i| = 0\} = 1 - (P\{v \notin D_i\})^n \tag{11}$$

where v is any specific vertex. The second equality follows from symmetry. But

$$\begin{aligned} P\{v \notin D_i\} &= P\{v \notin L_i \cap v \notin D_i\} + P\{v \in L_i \cap v \notin D_i\} \\ &= 1 - p + P\{v \in L_i \cap \{v \in L_1 \cup L_2 \cup \cdots \cup L_{i-1}\}\}. \end{aligned}$$

Since the choices of the elements of M are independent, the events $\{v \in L_i\}$ and $\{v \in L_1 \cup L_2 \cup \cdots \cup L_{i-1}\}$ are also independent. Hence

$$\begin{aligned} P\{v \notin D_i\} &= 1 - p + P\{v \in L_i\} P\{v \in L_1 \cup L_2 \cup \cdots \cup L_{i-1}\} \\ &= 1 - p + p\left(1 - (1-p)^{i-1}\right) = 1 - p(1-p)^{i-1}. \end{aligned}$$

By (11), $P\{|D_i| \geq 1\} = 1 - (1 - p(1-p)^{i-1})^n$. By (10),

$$E(|A_m|) = n(1-p)^m + \frac{1}{p}(1 - (1-p)^m) - \sum_{i=1}^{m} (1-p)^{m-i} \left(1 - p(1-p)^{i-1}\right)^n.$$

In the interesting case where $mp \geq \alpha \log n$ for some constant $\alpha > 1$ and $m \geq n$ (implying that $p \to 0$) we get

$$\begin{aligned} E(|A_m|) &\sim n(1-p)^m + \frac{1}{p}(1 - (1-p)^m) - \sum_{i=1}^{m} (1-p)^{m-i} (1-p)^n \\ &\sim \frac{1}{p}(1 - (1-p)^n). \end{aligned}$$

We now distinguish three cases, covering all possible values of np.

Table 1

Performance of Algorithms I, II and III, for $mp = \alpha \log n, p \to 0$

np	T_I	T_{II}, T_{III}	r_I	r_{II}	r_{III}
$np < \frac{1}{8\alpha \log n}$	$O(\frac{n \log n}{p})$	$O(\frac{n \log n}{p})$	$(\geq)\frac{8}{9}n$	$\frac{n}{4}$	$(1-\beta)n$
$np \to 0$	$O(\frac{n \log n}{p})$	$O(\frac{n \log n}{p})$			$(1-\beta)n$
$np \to b$	$O(\frac{n \log n}{p})$	$O(\frac{n \log n}{p})$	$(\sim)\frac{n}{\alpha b \log n}$		$(1-\beta)n(1-e^{-b})$
$np \to \infty$	$O(\frac{n \log n}{p})$	$O(\frac{n \log n}{p})$	$(\sim)\frac{1}{\alpha p \log n}$		$(1-\beta)\frac{n}{\log n}$
$np > \frac{1}{p}$	$O(n^2 p \log n)$	$O(\frac{n \log n}{p})$	$(\sim)\frac{1}{\alpha p \log n}$		$(1-\beta)\frac{n}{\log n}$

(1) If $np \to \infty$ then $E(|A_m|) \sim \frac{1}{p}$. The largest p to have $np \to \infty$, $mp \geq \alpha \log n$ and $m \geq n$ is $p = \frac{\log n}{n}$. So, we conclude that $E(|A_m|) = \Omega(\frac{n}{\log n})$.
(2) If $np \to b$ where $b > 0$ is a constant then $E(|A_m|) \sim \frac{n}{b}(1 - e^{-b}) = \Theta(n)$.
(3) If $np \to 0$ then $E(|A_m|) \sim \frac{1}{p}(1 - 1 + np) = \Theta(n)$.

The proof ends with the observation that since $E(|A_m|) \to \infty$ in all tree cases, then one can use Chernoff bounds to prove that $|A_m| \geq (1-\beta)E(|A_m|)$ for any constant $\beta > 0$ with very high probability.

□

Table 3.1 summarizes the performance of Algorithms I, II and III, in the case $mp = \alpha \log n$ and $p \to 0$. In the table, T_I, T_{II} and T_{III} denote the running times of algorithms I, II and III respectively. Also, r_I, r_{II} and r_{III} denote lower bounds on the sizes of the independent sets constructed (with high probability) by algorithms I, II and III respectively. The constants $\alpha > 1, \beta > 0$ and $b > 0$ are the same constants used in Theorem 7 and Theorem 9. Note that the running times of algorithms I, II and III are easily seen to be $O(mn + n + e(G)), O(nm)$ and $O(mn)$ respectively, but can be much smaller than these values (which correspond to the worst case running times), because they depend on the density of the graph. Also, due to lack of space in the table, we do not show the probabilities of success of each algorithm. It is worth mentioning that while the last two algorithms have a small probability of failure (that goes to 0 with n), algorithm I finds an independent set whose size is at least as much as the value given in the table, with probability 1.

4 Conclusions and Further Work

We proposed a very general, yet tractable, model for random intersection graphs. We believe that it can be useful in many technological applications.

We also did the first step in analysing algorithms for such graphs, and for the problem of construction of large independent sets of vertices. The finding of efficient algorithms for other interesting graph objects (e.g. long paths, giant components, dominating sets etc) is a subject of our future work.

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