Recent Advances in Multiobjective Optimization*

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Multiobjective (or multicriteria) optimization is a research area with rich history and under heavy investigation within Operations Research and Economics in the last 60 years [1,2]. Its object of study is to investigate solutions to combinatorial optimization problems that are evaluated under several objective functions – typically defined on multidimensional attribute (cost) vectors. In multiobjective optimization, we are interested not in finding a single optimal solution, but in computing the *trade-off* among the different objective functions, called the *Pareto set (or curve)* $\mathcal{P}$, which is the set of all feasible solutions whose vector of the various objectives is *not* dominated by any other solution.

Multiobjective optimization problems are usually NP-hard due to the fact that the Pareto set is typically exponential in size (even in the case of two objectives). On the other hand, even if a decision maker is armed with the entire Pareto set, s/he is still left with the problem of which is the "best" solution for the application at hand. Consequently, three natural approaches to deal with multiobjective optimization problems are to:

(i) Study approximate versions of the Pareto curve that result in (guaranteed) near optimal but smaller Pareto sets.
(ii) Optimize one objective while bounding the rest (*constrained approach*).
(iii) Proceed in a normative way and choose the "best" solution by introducing a utility (often non-linear) function on the objectives (*normalization approach*).

Until quite recently, the vast majority of research in multiobjective optimization [1,2] had focussed either on exact methods (i.e., to compute the entire Pareto set), or approximation methods through heuristic and metaheuristic approaches (that do not provide guarantees on the quality of the returned solution). An important outcome of the existing literature is that the two objectives case has been extensively studied, while there is a certain lack of efficient (generic) methods for the case of more than two objectives. Most importantly, there has been

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a lack of a systematic study of the complexity issues regarding approximate versions of the Pareto set (in a way analogous to the well-established approximation theory for single objective optimization problems).

The normalization approach has been quite investigated, especially within Operations Research. In this approach, a utility function is introduced that translates (in a linear or non-linear way) the different criteria into a common cost measure. For instance, when travelling in a traffic network one typically wishes to minimize travel distance and time; both criteria can be translated into a common cost measure (e.g., money), where the former is linearly translated, while the latter non-linearly (small amounts of time have relatively low value, while large amounts of time are very valuable). Under the normalization approach, we seek for a single optimum in the Pareto set (a feasible solution that optimizes the utility function). Due to the exponential size of the Pareto set, a fair portion of research had focussed on solving relaxed versions of the optimization problem, which corresponds to finding the best solution in the convex hull of the Pareto set. This turns out to be a good starting point to locate the exact solution by applying heuristic methods. However, the approaches used so far employ exhaustive algorithms for solving the relaxations of the problem at hand with complexities bounded by some polynomial in the size of the convex hull (which can be subexponentially large). In a very recent study [9], the first efficient (polynomial time) algorithm for solving the relaxation of the normalized version of the bicriteria shortest path is given, when the utility function is non-linear and convex.

The constrained approach had been (almost exclusively) the method adopted within Computer Science to deal with multiobjective optimization problems; see e.g., [3,4,6,8]. Classical examples concern the restricted (or constrained) shortest path and the restricted (or constrained) spanning tree problems.

Very recently, a systematic study (within Computer Science) has been initiated regarding the complexity issues of approximate Pareto curves [7]. Informally, an $(1+\varepsilon)$-*Pareto curve* $\mathcal{P}_\varepsilon$ is a subset of feasible solutions such that for any Pareto optimal solution and any $\varepsilon > 0$, there exists a solution in $\mathcal{P}_\varepsilon$ that is no more than $(1+\varepsilon)$ away in all objectives. Although this concept is not new (it has been previously used in the context of bicriteria and multiobjective shortest paths [5,12]), Papadimitriou and Yannakakis in a seminal work [7] show that for *any* multiobjective optimization problem there exists a $(1+\varepsilon)$-Pareto curve $\mathcal{P}_\varepsilon$ of (polynomial) size $|\mathcal{P}_\varepsilon| = O((4B/\varepsilon)^{d-1})$, where B is the number of bits required to represent the values in the objective functions (bounded by some polynomial in the size of the input). They also provide a necessary and sufficient condition for its efficient (polynomial in the size of the input and $1/\varepsilon$) construction. In particular, $\mathcal{P}_\varepsilon$ can be constructed by $O((4B/\varepsilon)^d)$ calls to a GAP routine that solves (in time polynomial in the size of the input and $1/\varepsilon$) the following problem: given a vector of values $\mathbf{a}$, either compute a solution that dominates $\mathbf{a}$, or report that there is no solution better than $\mathbf{a}$ by at least a factor of $1+\varepsilon$ in all objectives. Extensions to that method to produce a constant approximation to the smallest possible $(1+\varepsilon)$-Pareto curve for the cases of 2 and 3 objectives are presented in [11], while for $d > 3$ objectives inapproximability results are shown for such a constant approximation.

Apart from the above general results, there has been very recent work on improved approximation algorithms (FPTAS) for multiobjective shortest paths [10]. In that paper, a new and remarkably simple algorithm is given that constructs $(1 + \varepsilon)$-Pareto sets for the single-source multiobjective shortest path problem, which improves considerably upon previous approaches. In the same paper, it is also shown how this algorithm can provide better approximation schemes for both the constrained and the normalized versions of the problem for any number of objectives. An additional byproduct is a generic method for constructing FPTAS for any multiobjective optimization problem with non-linear objectives of a rather general form (that includes any polynomial of bounded degree with non-negative coefficients). This method does not require the existence of a GAP routine for such non-linear objectives.

All these very recent algorithmic and complexity issues will be discussed and elaborated in the talk.

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