Probabilistic Techniques

Homework

Problem 1 (2.5)

Prove that, for every integer \( n \), there exists a coloring of the edges of the complete graph \( K_n \) by two colours so that the total number of monochromatic \( K_4 \) subgraphs is at most \( \binom{n}{4} 2^{-5} \).

Problem 2 (2.5)

Suppose \( n > 4 \) and let \( H \) be an \( n \)-uniform hypergraph with at most \( 4^{n-1}/3^n \) edges. Prove that there is a coloring of the vertices of \( H \) by 4 colors so that in every edge all 4 colors are represented.

Problem 3 (3.0)

Find the threshold probability for the existence with high probability of paths of length 2 in \( G_{n, \frac{1}{2}} \).

Problem 4 (1.0)

Prove that if there is a real \( p \), \( 0 \leq p \leq 1 \), so that
\[
\binom{n}{k} p^k (1-p)^{n-k} + \binom{n}{t} (1-p)^t < 1,
\]
then the Ramsey number \( r(k, t) \) satisfies \( r(k, t) > n \).

Problem 5 (1.0)

Prove that there exists a two-coloring of the edges of the complete bipartite graph \( K_{m,n} \) with at most \( \binom{m}{a} \binom{n}{b} 2^{1-ab} + \binom{n}{a} \binom{m}{b} 2^{1-ab} \) monochromatic \( K_{a,b} \).