Lecture 2: "A Las Vegas Algorithm for finding the closest pair of points in the plane"

Sotiris Nikoletseas
Professor

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Las Vegas algorithms

Definition: A Las Vegas algorithm is a randomized algorithm that always returns the correct result.

However, its running time may change, since this time is actually a random variable.

The closest pair of points problem

Definition: Given a set of points P in the plane, find the pair of points closest to each other. Formally, return the pair of points, realizing (the closest possible inter-point distance):

$$CP(P) = \min_{p,q \in P} \|pq\|$$

where $\|pq\|$ denotes the Euclidean distance of points p, q.

Note: The problem can naively be solved in $O(n^2)$ time, by computing all $\binom{n}{2}$ inter-point distances.

Here, we will present a Las Vegas algorithm of $O(n)$ expected time.

The grid G_r - I

- For r positive and a point $p = (x, y)$ in R^2, let $G_r(p)$ the point $\left(\left\lfloor \frac{x}{r} \right\rfloor, \left\lfloor \frac{y}{r} \right\rfloor\right)$
  e.g. $p = (4.5, 7.6)$ and $r = 2 \Rightarrow G_2(p) = (2, 3)$

- We call r the width of grid G_r.

- Actually, the grid G_r partitions the plane into square regions, which we call grid cells. Formally, a grid cell is defined, for $i, j \in \mathbb{Z}$, by the intersection of the four half-planes: $x \geq ri, x < (r+1)i, y \geq rj, y < (r+1)j$

The grid G_r - II

- The partition of points in P into subsets by the grid G_r is denoted by $G_r(P)$. Formally, two points $p, q \in P$ belong into the same set of the $G_r(P)$ partition iff they belong into the same grid cell. Equivalently, they are mapped into the same grid point $G_r(p) = G_r(q)$.

- We call a block of continuous grid cells a grid cluster.

A data structure for the grid

- Note: every grid cell C of G_r has a unique ID. Indeed, let $p = (x, y)$ be any point in cell C and consider $id_p = \left(\left\lfloor \frac{x}{r} \right\rfloor, \left\lfloor \frac{y}{r} \right\rfloor\right)$, which is actually the unique ID id_c of cell C, since only points in the cell C are mapped to id_c.
- This allows an efficient storage of the set P of points inside a grid, as follows:
  (1) given a point p, we compute id_p
  (2) for each unique id (corresponding to a cell) we maintain a linked list of all the points in that cell
  (3) we can thus fetch the data (the points) for a cell by hashing, in constant time.
  (i.e. we store pointers to all those linked lists in a hash table, where each list is indexed by its unique id).

An intermediate decision problem

We will employ the following intermediate result.

<u>Lemma 1:</u> Given a set P of n points in the plane, and a distance r, one can check in linear time whether $CP(P) < r$ or $CP(P) \geq r$

<u>Proof:</u>

- We store the points of P in the grid G_r (i.e. for every non-empty grid cell we maintain a linked list of the points inside it)
- Thus, adding a new point p takes constant time (compute $id(p)$, check if $id(p)$ already exists in the hash table; if it exists just add p to it; otherwise, create a new linked list for the cell with this ID and store p in it)
- Totally (for all n points) this will take $O(n)$ time.

An intermediate decision problem (continued)

Note: If any grid cell in $Gr(P)$ contains more than, say, 9 points of P, then $CP(P) < r$.

Indeed:

- Consider a cell C with more than 9 points of P
- Partition C into 3x3 equal squares
- Clearly, one of these 9 squares must contain two (or more) points of P and let C' this square
- The diameter of $C' = diam(C') = \frac{diam(C)}{3} = \frac{\sqrt{r^2+r^2}}{3} < r$
- Thus, at least two points of P in C' are at distance smaller than r from each other

Note: The 9 points argument is indicative (e.g. we could consider 16 points and partition the cell into 4x4 equal squares).

Proof of decision lemma 1 (continued)

- Thus, when we insert a point p, we can fetch all P points already inserted, for the cell of p, as well as its 8 adjacent cells
- All those cells must contain at most 9 points of P each (otherwise we would have stopped knowing that $CP(P) < r$)
- Let S the set of all those points, so $|S| \leq 9 \cdot 9 = \Theta(1)$
- Thus, we can compute by brute force in $O(1)$ time the closest point to p in S. If its distance to p is $< r$ then we stop (with $CP(P) < r$); otherwise we continue with the other (at most) 80 points
- Overall this takes $O(n)$ time.

(end of Lemma 1 proof)

An intuitive way of computing $CP(P)$

- Permute arbitrarily the points in P
- Let $P = \langle p_1, \ldots, p_n \rangle$ the resulting permutation
- Let $\underline{r_{i-1} = CP(\{p_1, \ldots, p_{i-1}\})}$ i.e., the "partial knowledge" of $CP(P)$ after exposing the first $i-1$ points of the permutation ($P_{i-1} = \langle p_1, \ldots, p_{i-1} \rangle$)
- We <u>check whether $r_i < r_{i-1}$</u> by calling the algorithm of Lemma 1 on P_i and r_{i-1}

<u>NOTE:</u> A grid G_r can only answer (via Lemma 1) queries of the type $CP(P) < r$,
while for finer queries $CP(P) < r' < r$ a finer granularity grid must be rebuilt!

Computing $CP(p)$ (continued)

Thus, when "exposing" one more point (i.e. going from $P_{i-1} = \langle p_1, \ldots, p_{i-1} \rangle$ to $P_i = \langle p_1, \ldots, p_{i-1}, p_i \rangle$ we distinguish two different cases:

- THE BAD CASE: If $r_i < r_{i-1}$ a new, finer granularity grid G_{r-1} must be built, and insert points $p_1, \ldots, p_i$ to it. This takes obviously $O(i)$ time.
- THE GOOD CASE: If $r_i = r_{i-1}$, i.e. the distance of the closest pair does not change by adding p_i. In this case, we do not need to rebuild the grid and inserting the new point p_i takes constant time.

Intuitive remark on time complexity

- No change in closest pair distance after a point insertion $\Rightarrow$ constant time needed
- A change after inserting point $i \Rightarrow O(i)$ time needed (to rebuild the data structure)
- If the closest pair distance never changes $\Rightarrow O(1)$ cost n times $\Rightarrow O(n)$ time needed
- If it changes all the time $\Rightarrow O\left(\sum_{i=3}^{n} i\right) = O(n^2)$ time
- If it changes K times $\Rightarrow$ in the worst case $O(Kn)$ time needed

Expected linear time

Lemma 2: Let P a set of n points in the plane. One can compute the closest pair of them in expected linear time.

Proof:

- Randomly permute the points of P into $P_n = \langle p_1, \ldots, p_n \rangle$
- Let $r_2 = \|p_1 p_2\|$ and start inserting points to the data structure based on Lemma 1
- If at the ith iteration $r_i = r_{i-1} \Rightarrow$ addition of p_i takes constant time
- If $r_i < r_{i-1}$ then rebuild the grid, and reinsert the i points in $O(i)$ time
- Let X_i a random indicator variable:

$$X_i = \begin{cases} 1, r_i < r_{i-1} \\ 0, r_i = r_{i-1} \end{cases}$$

Proof of Lemma 2

- Let T the running time of the method. Clearly
  $X = 1 + \sum_{i=2}^{n} (1 + X_i \cdot i)$
- By linearity of expectation it is:
  $E(X) = E\left[1 + \sum_{i=2}^{n} (1 + X_i \cdot i)\right] = 1 + \sum_{i=2}^{n} E\left(1 + X_i \cdot i\right) =$
  $1 + \sum_{i=2}^{n} 1 + \sum_{i=2}^{n} E\left(X_i \cdot i\right)$
  But
  $E\left(X_i \cdot i\right) = 0 \cdot Pr\{X_i = 0\} + i \cdot Pr\{X_i = 1\} = i \cdot Pr\{X_i = 1\}$
  Thus
  $E(X) = 1 + n - 1 + \sum_{i=2}^{n} i \cdot Pr\{X_i = 1\} = n + \sum_{i=2}^{n} i \cdot Pr\{X_i = 1\}$

Bounding the probability of a change ($Pr\{X_i = 1\}$)

We will bound $Pr\{X_i = 1\} = Pr\{r_i < r_{i-1}\}$

- Fix the points of $P_i = \{p_1, p_2, \ldots, p_i\}$
- Randomly permute these points

Definition: A point $q \in P_i$ is called critical (at phase i) if

$$CP(P_i \backslash \{q\}) > CP(P_i)$$

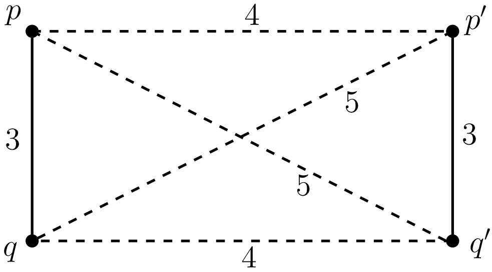
i.e. if its "consideration" leads to a"change" (e.g. smaller inter point closest distance)

Note:

1. Whether a node, at a given phase, is critical or not, only depends on geometry (not on the order that the algorithm examines the points). The order only affects the probability of a change or not.
2. The notion of criticality refers to a given phase of algorithm evolution.
3. There are 3 cases: 0, 1 or 2 critical points.

Bounding the probability of a change ($Pr\{X_i = 1\}$)

Case 1: no critical points



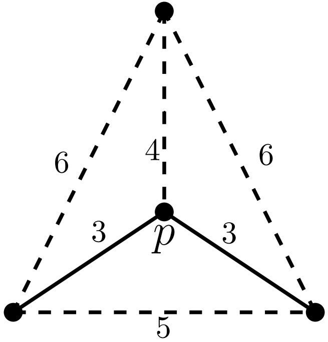
If there are *no* critical points $\Rightarrow r_i = r_{i-1} \Rightarrow$ no change

$$\Rightarrow Pr\{X_i = 1\} = 0$$

(When more than one, vertex disjoint closest distance edges exist.)

Bounding the probability of a change ($Pr\{X_i = 1\}$)

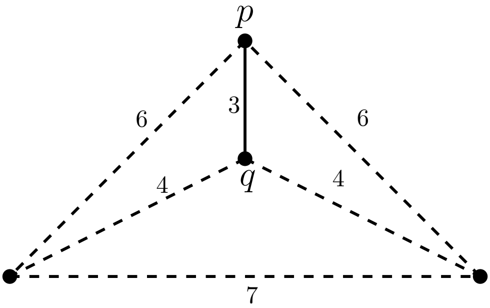
Case 2: 1 critical point



If there is *one* critical point $\Rightarrow Pr\{X_i = 1\} = \frac{1}{i}$
(this is the probability that p_i is last in permutation)
(When a single set of more than one, adjacent closest pair edges exists.)

Bounding the probability of a change ($Pr\{X_i = 1\}$)

Case 3: 2 critical points



If there are *two* critical points, let them p, q and notice that this is the unique points pair realizing $CP(P_i)$. But then $r_i < r_{i-1}$ iff either p or q are the last point (p_i) in the permutation, an event with probability $\frac{2}{i}$
(When a single closest distance edge exists.)

Bounding the change of probability

Case >2 critical points

Note that there cannot be more than two critical points.

Proof:
Indeed, let p and q be critical (and realize $CP(P)$). Let now a third critical point r. Then it must be $CP(P_i \backslash r) > CP(P_i)$. But, $CP(P_i \backslash r) = \|pq\|$ (since if we exclude r then the closest distance is the one of the p, q critical points). But $\|pq\| = CP(P_i) \Rightarrow CP(P_i) > CP(P_i)$, a contradiction. $\square$

Concluding the expected time analysis

Thus,

$$E[T] = n + \sum_{i=2}^{n} i Pr\{X_i = 1\} \leq n + \sum_{i=2}^{n} i \frac{2}{i} =$$
$$= n + \sum_{i=2}^{n} 2 = n + 2n - 2 < 3n$$

Overall, the expected running time is $O(n)$ i.e., linear.