Lecture 4: "Randomized selection"

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1. Preliminaries

(i) Boole's inequality (or union bound)

Let random events $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$. Then

$$Pr\left\{\bigcup_{i=1}^{n} \mathcal{E}_i\right\} = Pr\{\mathcal{E}_1 \cup \mathcal{E}_2 \cup \cdots \cup \mathcal{E}_n\} \leq \sum_{i=1}^{n} Pr\{\mathcal{E}_i\}$$

Note: If the events are disjoint, then we get equality.

(ii) Expectation (or Mean)

Let X a random variable with probability density function (pdf) $f(x)$. Its expectation is:

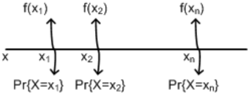
$$\mu_x = E[X] = \sum_x x \cdot Pr\{X = x\}$$

If X is continuous, $\mu_x = \int_{-\infty}^{\infty} x f(x)\, dx$

(ii) Expectation (or Mean)

Properties:

- $\forall X_i \ (i = 1, 2, \ldots, n) : E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$
  This important property is called "linearity of expectation".
- $E[cX] = cE[X]$, where c constant
- if X, Y stochastically independent, then $E[X \cdot Y] = E[X] \cdot E[Y]$
- Let $f(X)$ a real-valued function of X. Then $E[f(x)] = \sum_x f(x) Pr\{X = x\}$



1. Preliminaries

(iii) Markov's inequality

Theorem: Let X a non-negative random variable. Then, $\forall t > 0$

$$Pr\{X \geq t\} \leq \frac{E[X]}{t}$$

Proof: $$E[X] = \sum_x x Pr\{X = x\} \geq \sum_{x \geq t} x Pr\{X = x\}$$

$$\geq \sum_{x \geq t} t Pr\{X = x\} = t \sum_{x \geq t} Pr\{X = x\} = t \; Pr\{X \geq t\}$$

Note: Markov is a (rather weak) concentration inequality, e.g.

$Pr\{X \geq 2E[X]\} \leq \frac{1}{2}$

$Pr\{X \geq 3E[X]\} \leq \frac{1}{3}$

etc

(iv) Variance (or second moment)

- Definition: $Var(X) = E[(X-\mu)^2]$, where $\mu = E[X]$
  i.e. it measures (statistically) deviations from mean.

- Properties:
  - $Var(X) = E[X^2] - E^2[X]$
  - $Var(cX) = c^2 Var(X)$, where c constant.
  - if X, Y independent, it is $Var(X+Y) = Var(X) + Var(Y)$

Note: We call $\sigma = \sqrt{Var(X)}$ the standard deviation of X.

(v) Chebyshev's inequality

Theorem: Let X a r.v. with mean $\mu = E[X]$. It is:

$$Pr\{|X-\mu| \geq t\} \leq \frac{Var(X)}{t^2} \qquad \forall t > 0$$

Proof: $Pr\{|X-\mu| \geq t\} = Pr\{(X-\mu)^2 \geq t^2\}$

From Markov's inequality:

$$Pr\{(X-\mu)^2 \geq t^2\} \leq \frac{E[(X-\mu)^2]}{t^2} = \frac{Var(X)}{t^2}$$

Note: Chebyshev's inequality provides stronger (than Markov's) concentration bounds, e.g.

$Pr\{|X-\mu| \geq 2\sigma\} \leq \frac{1}{4}$

$Pr\{|X-\mu| \geq 3\sigma\} \leq \frac{1}{9}$

etc

2. The Randomized Selection Algorithm

- The problem: We are given a set S of n distinct elements (e.g. numbers) and we are asked to find the kth smallest.
- Notation:
  - $r_S(t)$: the rank of element t (e.g. the smallest element has rank 1, the largest n and the kth smallest has rank k).
  - $S_{(i)}$ denotes the ith smallest element of S (clearly, we seek $S_{(k)}$ and $r_S(S_{(k)}) = k$).
- Remark: the fastest known deterministic algorithm needs $3n$ time and is quite complex. Also, any deterministic algorithm requires $2n$ time (a tight lower bound).

2. The basic idea: random sampling

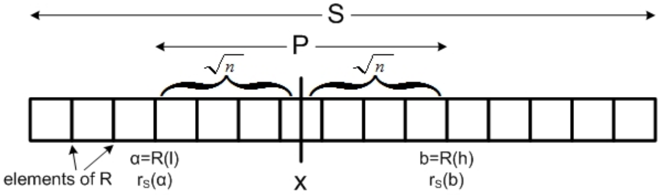
- we will randomly sample a subnet of elements from S, trying to optimize the following trade-off:
  - the sample should be small enough to be processed (e.g. ordered) in small time
  - the sample should be large enough to contain the kth smallest element, with high probability

2. The Lazy Select Algorithm

1. Pick randomly uniformly, with replacement, a subset R of $n^{\frac{3}{4}}$ elements from S.
2. Sort R using an optimal deterministic sorting algorithm.
3. Let $x = k \cdot n^{-\frac{1}{4}}$.
   $l = max\{\lfloor x - \sqrt{n} \rfloor, 1\}$ and $h = min\{\lceil x + \sqrt{n} \rceil, n^{\frac{3}{4}}\}$.
   $a = R_{(l)}$ and $b = R_{(h)}$
   By comparing a and b to every element of S, determine $r_S(a), r_S(b)$.
4. If $k \in [n^{\frac{1}{4}}, n - n^{\frac{1}{4}}]$, let $P = \{y \in S : a \leq y \leq b\}$.
   Check whether $S_{(k)} \in P$ and $|P| \leq 4n^{\frac{3}{4}} + 2$. If not, repeat steps 1-3 until such a P is found.
5. By sorting P, identify $P_{(k - r_S(a) + 1)} = S_{(k)}$.

2. Remarks on the Lazy Select Algorithm

- In Step 1, sampling is done with replacement to simplify the analysis. Sampling without replacement is marginally faster but more complex to implement.
- Step 2 takes $O(n^{\frac{3}{4}} \log n)$ time (which is $o(n)$).
- Step 3 clearly takes $2n$ time ($2n$ comparisons). Graphically,



An example: assume $r_S(a) = 3$ and we want $S_{(7)}$. In the sorted list of P elements, $S_{(7)} = P_{(k - r_S(a)+1)} =$
$= P_{(7-3+1)} = P_5$, i.e. the 5th element indeed.

2. Remarks on the Lazy Select Algorithm

- In Step 4, it is easy to check (in constant time) whether $S_{(k)} \in P$ by comparing k to (the now known) $r_S(a), r_S(b)$.
- In Step 5, sorting P takes $O(n^{\frac{3}{4}} \log n) = o(n)$ time.

Note: we skip in Step 4 the (less interesting) cases where $k < n^{\frac{1}{4}}$ and $k > n - n^{\frac{1}{4}}$. Their analysis is similar.

2. When Lazy Select fails?

The algorithm may fail in Step 4, either because $\underline{S_{(k)} \notin P}$ because $\underline{|P| \text{ is large}}$. We will show that the probability of failure is very small.

Lemma 1. The probability that $S_{(k)} \notin P$ is $O(n^{-\frac{1}{4}})$.

Proof: This happens if i)$S_{(k)} < a$ or ii)$S_{(k)} > b$.

i) $S_{(k)} < a \Leftrightarrow$ fewer than l ($l = k \cdot n^{-\frac{1}{4}} - \sqrt{n}$) of the samples in R are less than or equal to $S_{(k)}$. Let:

$$X_i = \begin{cases} 1, & \text{the } i\text{th random sample is at most } S_{(k)} \\ 0, & \text{otherwise} \end{cases}$$

Clearly, $E(X_i) = Pr\{X_i\} = \frac{k}{n}$ and $Var(X_i) = \frac{k}{n}(1 - \frac{k}{n})$

Let $X = \sum_{i=1}^{|R|} X_i = \#$ samples in R that are at most $S_{(k)}$. Then

2. When Lazy Select fails?

$\mu_X = E[X] = |R| \cdot E[X_i] = n^{\frac{3}{4}} \frac{k}{n} = kn^{-\frac{1}{4}}$ and

$\sigma_X^2 = Var[X] = \sum_{i=1}^{|R|} Var(X_i) = n^{\frac{3}{4}} \frac{k}{n}(1 - \frac{k}{n}) \leq \frac{n^{\frac{3}{4}}}{4}$ (since the samples are independent)

Thus, $Pr\{|X - \mu_X| \geq \sqrt{n}\} \leq \frac{\sigma_X^2}{n} \leq \frac{n^{\frac{3}{4}}}{4n} = O(n^{-\frac{1}{4}})$
$\Rightarrow Pr\{X - \mu_X < -\sqrt{n}\} \leq O(n^{-\frac{1}{4}})$
$\Rightarrow Pr\{X < \mu_X - \sqrt{n}\} = Pr\{X < \underbrace{kn^{-\frac{1}{4}} - \sqrt{n}}_{l}\} \leq O(n^{-\frac{1}{4}})$

2. When Lazy Select fails?

ii) The case $S_{(k)} > b$ is essentially symmetric (at least h of the random samples should be smaller than $S_{(k)}$), so

$$Pr\{S_{(k)} > b\} = O(n^{-\frac{1}{4}})$$

Overall $Pr\{S_{(k)} \notin P\} = Pr\{S_{(k)} < a \cup S_{(k)} > b\} =$
$O(n^{-\frac{1}{4}}) + O(n^{-\frac{1}{4}}) = O(n^{-\frac{1}{4}})$ $\square$

2. The Lazy Select Algorithm

Lemma 2 The probability that P contains more than $4n^{\frac{3}{4}} + 2$ elements is $O(n^{-\frac{1}{4}})$

Proof: Very similar to the proof of Lemma 1: Let

$$k_e = max\{1, k - 2n^{\frac{3}{4}}\} \text{ and}$$
$$k_n = min\{k + 2n^{\frac{3}{4}}, n\}$$

If $S_{(k_l)} < a$ or $S_{(k_h)} > b$ then P contains more than $4n^{\frac{3}{4}} + 2$ elements. For simplicity, let $k_l = k - 2n^{\frac{3}{4}}, k_h = k + 2n^{\frac{3}{4}}$
Then, it suffices to "simulate" the proof of Lemma 1 for $k = k_l$ and then for $k = k_h$.

2. The Lazy Select Algorithm

Theorem The Algorithm Lazy Select finds the correct solution with probability $1 - O(n^{-\frac{1}{4}})$ performing $2n + o(n)$ comparisons.

Proof: Due to Lemmata 1, 2 the Algorithm finds $S_{(k)}$ on the first pass through steps 1-5 with probability $1 - O(n^{-\frac{1}{4}})$ (i.e., it does not fail in Step 4 avoiding a loop to Step 1). Step 1 obviously takes $o(n)$ time. Step 2 requires $O(n^{\frac{3}{4}} \log n) = o(n)$ time, and Step 3 clearly needs $2n$ comparisons (comparing each of the n elements of S to a and b). Overall the time needed is thus $2n + o(n)$.