Course “Algorithmic Foundations of Sensor Networks”
Lecture 3: Energy-efficient and robust routing

Sotiris Nikoletseas
Associate Professor

Department of Computer Engineering and Informatics
University of Patras, Greece

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Routing can be made robust and energy efficient by taking into account some pieces of *state information available locally* in the network, such as:

- **link quality**: e.g. packet reception rates obtained by periodic monitoring
- **link distance**: if link monitoring is problematic/expensive (e.g. in highly dynamic, rapidly fading environments) link distances can be alternatively used as *indicators* of link quality and energy consumption
- **residual energy**: avoid nodes with low energy, to prolong the network lifetime (Lecture 4)
- **location information**: if such information is available, geographic routing techniques can be used (Lecture 8)
- **mobility information** can also be exploited; sensor/sink mobility is both a challenge and an opportunity (Lecture 9)
Overview

A. Link quality based methods (ETX, MOR)
B. Link distance based methods
   - Routing with relay diversity (ExOR)
   - Greedy, local neighbor selection (LTP)
C. Multipath routing
   - Braided paths, GRAd, GRAB
   - Probabilistic forwarding (PFR)
A. Link Quality Methods

- **Main goal**: Providing robustness via selecting routes that minimize end-to-end retransmissions or failure probabilities.

- **Motivation**: If wireless links were (ideally) error-free, then a shortest hop-count path could be chosen. Such paths however require high distance links, which in realistic settings are likely to be error-prone. Thus, the link quality should be taken into account (via simple ACK signals for each successfully delivered packet).
A1. The ETX metric

- Let $d_f$ be the packet reception rate (probability of successful delivery) over a link in the forward direction, and $d_r$ the probability that the corresponding ACK is received in the reverse direction.
- Assuming each packet transmission can be abstracted as a Bernoulli trial, the expected number of retransmissions required for successful delivery of a packet on this link is:

$$ETX = \frac{1}{d_f \cdot d_r}$$

- This metric for a single link can then be incorporated into any routing protocol, so that *chosen end-to-end paths minimize the sum of ETX over all links on the path*, i.e. the total expected number of transmissions on the path.
This example shows 3 routes from A to B, with the forward probabilities on each link (for simplicity, all reverse probabilities are taken $d_r = 1$).

- The direct A-B transmission would require an expected number of 10 retransmissions.
- The path via C,D,E would require $4(1/0.9) \approx 4.44$ retransmissions (1.1 per link) on average.
- The A-F-B path would require $2(1/0.8) \approx 2.5$ retransmissions (1.25 per link) on average.

⇒ the ETX-minimizing path is A-F-B.
• As demonstrated in the example, ETX *neither favors long paths* with many short-distance, high-quality links, *nor very short paths* with a few long-distance, low-quality links. It actually takes an “in-between” approach.

• **ETX advantages**: It minimizes the number of transmissions required, as well as it improves energy efficiency. Also, it directly addresses the potential link asymmetries, by taking into account packet delivery probabilities in both directions.

• **ETX limitation**: It assumes knowledge of the $d_r, d_f$ packet reception probabilities. In quite static networks these values may be periodically obtained via link monitoring, however in dynamic networks the obtained values may become obsolete very soon and the link monitoring overhead may be prohibitively high.
• Under high dynamics (mobile nodes, mobile objects) the link quality fluctuates very rapidly and ETX is not feasible. In such cases, analytic models of link quality can be used. They explicitly model the wireless channel as having multi-path fading with Rayleigh statistics (fluctuations over time).

• Let $d$ the distance between transmitter and receiver, $h$ the path-loss exponent, SNR the signal-to-noise ratio without fading, $f$ the fading state of the channel. Then, the instantaneous capacity of the channel is:

$$ C = \log \left( 1 + \frac{|f|^2}{d^h \cdot SNR} \right) $$

• The outage probability $P_{out}$ is defined as the probability that the instantaneous channel capacity falls below the transmission rate $R$. It is shown that:

$$ P_{out} = 1 - \exp \left( \frac{-d^h}{\mu \cdot SNR^*} \right) $$

where $\mu = E[|f|^2]$ is the mean of the Rayleigh fading and $SNR^*$ a normalized SNR.
Then, end-to-end reliability of a route is defined as the probability that *none of the intermediate links suffers outage*, and the most reliable route between two nodes is one that *minimizes the following path metric*:

$$
\sum_i d^h_i
$$

where $d_i$ is the distance of the $i^{th}$ hop in the path. The metric $d^h_i$ for each link is called the minimum outage route (MOR) metric.

The MOR approach does not require collection of link quality metrics like ETX, neither ACK messages are used. However, *its practicality is limited by the basic abstraction that the channel fading follows a certain distribution* (the Rayleigh distribution).
B. Link Distance Methods

- In general, nodes *closer to the data destination* are chosen/favored.
- The relay diversity provided by the fact that wireless transmissions are *broadcast to multiple nodes* is exploited.
• An example:

In traditional routing, the reliability of the AC path depends on whether both AB and BC transmissions have been successful.

However, if C is also allowed to accept packets directly from A, then the reliability can be further increased without much additional energy cost.

Allowing such 2-hop packet reception, in the high SNR regime, the end-to-end outage probability decays as \((SNR)^{-2}\). When nodes within \(L\) hops can communicate with each other with high SNR, this probability can become as small as \((SNR)^{-L}\).

A weakness of this method is that it requires a larger number of receivers to be actively overhearing each message, which may incur a radio energy penalty.
Main idea: the identity of the node which will eventually forward a packet, is not predetermined before the packet is transmitted. Instead, the method tends to ensure that the node closest to the destination that receives a given packet will forward the packet further.

The protocol has three stages:

1. **Priority ordering**: At each step, the transmitter includes in the packet a schedule with the *priority order* candidate receivers should forward the packet.
2. *Transmission acknowledgements:* A MAC scheme is used so that each candidate receiver sends the ID of the highest-priority successful recipient known to it. All nodes listen to all ACKs, so they distributively determine which node, among those who received the packet successfully, has highest priority.

3. *Forwarding decision:* After listening to all ACKs, the nodes that *have not* heard of any IDs with priorities greater than their own will transmit.

- **Features of the ExOR protocol**
  - *Nodes further away to the current node (yet closer to the destination)* are less likely to successfully receive the packet, but, whenever they receive it, they are favored to act as forwarders!
  - This *tends to make good progress towards the data destination,* without many transmissions and delays.
  - *As with relay diversity,* ExOR requires *a larger number of receivers to be active.* Also, the priority evaluation necessitates some inter-node packet delivery ratios to be tracked and maintained.
We call network nodes “particles”.

a) Each particle may have *two communication modes*: a *broadcast* (radio) *beacon mode* and a *directed to a point* transmission mode (laser beam). However, a radio broadcast suffices.

b) Each particle may alternate between a *sleeping* and an *awake* mode. During sleeping periods particles cease any communication.

c) Particles *do not move*.

d) The particles are spread in a *two-dimensional area* (plane).
e) A receiving wall $\mathcal{W}$ is a line in the plane. The wall represents the control center (multiple/mobile sinks).

f) Each particle is aware of the direction toward $\mathcal{W}$.

g) No geolocation abilities assumed

Definitions: Let $d$ (in numbers of particles / $m^2$) be the density of the cloud.

Let $\mathcal{R}$ be the maximum (beacon/laser) transmission range of each particle.
Let \( d(p_i, p_j) \) the (vertical) distance of \( p_i, p_j \) and \( d(p_i, \mathcal{W}) \) the (vertical) distance of \( p_i \) from \( \mathcal{W} \). Let \( \text{info}(\mathcal{E}) \) the info to be propagated. Each \( p' \) receiving \( \text{info}(\mathcal{E}) \) does the following:

- **Search Phase**: It uses a low energy broadcast of a beacon (angle \( \alpha \) above and below the vertical line) to discover a particle closer to \( \mathcal{W} \) (i.e. a \( p'' \) where \( d(p'', \mathcal{W}) < d(p', \mathcal{W}) \)).

- **Direct Transmission Phase**: If found, \( p' \) sends \( \text{info}(\mathcal{E}) \) to \( p'' \) via a direct line (laser) transmission.

- **Backtrack Phase**: If repetitions of the search phase fail to discover a particle nearer to \( \mathcal{W} \), then \( p' \) sends \( \text{info}(\mathcal{E}) \) to the particle it received the information from.
Example of the Search Phase

Example of Data Propagation
**Definitions:** Let $h_{\text{opt}}$ the (optimal) number of “hops” (vertical to $\mathcal{W}$ transmissions) needed to reach $\mathcal{W}$, if particles always exist in pair-wise distances $\mathcal{R}$ towards $\mathcal{W}$.

Let $h$ the actual number of hops (transmissions) taken to reach $\mathcal{W}$. The “hops” efficiency of the data propagation protocol is the ratio

$$C_h = \frac{h}{h_{\text{opt}}}$$

where $h_{\text{opt}} = \left\lceil \frac{d(p, \mathcal{W})}{\mathcal{R}} \right\rceil$
Why studying $h, C_h$?

When a particle $p$ “looks around” for a particle as close to $W$ as possible to pass information, it may not get any particle in the perfect direction (on the line vertical to $W$ passing from $p$), mainly because:

a) There might never have been any particles in that direction.

b) Particles of sufficient remaining battery power may not be available anymore.

c) Particles available may temporarily “sleep” to save energy.
The search phase always finds a $p''$ in the semicircle of center $p'$ and radius $R$ towards $W$. This assumption can be relaxed: (a) by repetitions of the search phase (b) we may consider a cyclic sector defined by circles of radii $R - \Delta R, R$ (c) if a search phase ultimately fails, the protocol backtracks.

- The position of $p''$ is random uniform in the arc of angle $2a$.
- Each target selection is stochastically independent of the others.
Lemma

The expected “hops” efficiency of LTP in the $\alpha$-uniform case is $E(C_h) \sim \frac{\alpha}{\sin \alpha}$, for large $h_{opt}$. Also, $1 \leq E(C_h) \leq \frac{\pi}{2} \approx 1.57$.

Proof: A sequence of points is generated, $p_0 = p$, $p_1$, $p_2$, ... $p_{h-1}$, $p_h$ where $p_{h-1}$ is the first particle found within $\mathcal{W}$’s range and $p_h$ is beyond $\mathcal{W}$. Let $\alpha_i$ be the (positive or negative) angle of $p_i$ w.r.t. $p_{i-1}$’s vertical line to $\mathcal{W}$. It is:

$$\sum_{i=1}^{h-1} d(p_{i-1}, p_i) \leq d(p, \mathcal{W}) \leq \sum_{i=1}^{h} d(p_{i-1}, p_i)$$
The (vertical) progress toward \( \mathcal{W} \) is \( \Delta_i = d(p_{i-1}, p_i) = R \cos \alpha_i \). We get:

\[
\sum_{i=1}^{h-1} \cos \alpha_i \leq h_{opt} \leq \sum_{i=1}^{h} \cos \alpha_i
\]

From Wald’s equation, then

\[
E(h-1) \cdot E(\cos \alpha_i) \leq E(h_{opt}) \leq E(h) \cdot E(\cos \alpha_i) \Rightarrow
\]

\[
\frac{\alpha}{\sin \alpha} \leq \frac{E(h)}{h_{opt}} = E(C_h) \leq \frac{\alpha}{\sin \alpha} + \frac{1}{h_{opt}}
\]

since

\[
E(\cos \alpha_i) = \int_{-\alpha}^{\alpha} \cos x \cdot \frac{1}{2\alpha} \cdot dx = \frac{\sin \alpha}{\alpha}
\]

Assuming large values for \( h_{opt} \) and since for \( 0 \leq \alpha \leq \frac{\pi}{2} \) it is \( 1 \leq \frac{\alpha}{\sin \alpha} \leq \frac{\pi}{2} \) we get the result.
The "min-two uniform targets" (M2TP) Protocol

We assume that the search returns two points $p''$, $p'''$ each uniform in $(-\alpha, \alpha)$ and that the protocol selects the best. Let $\alpha_{i1}$, $\alpha_{i2}$ the angles of the particles found and let $\alpha_i = \min \{|\alpha_{i1}|, |\alpha_{i2}|\}$. Then,

$$P\{\alpha_i > \phi\} = P\{|\alpha_{i1}| > \phi \cap |\alpha_{i2}| > \phi\} = \left(\frac{\alpha - \phi}{\alpha}\right)^2$$

Thus, the distribution function of $\alpha_i$, is

$$F_{\alpha_i}(\phi) = P\{\alpha_i \leq \phi\} = \frac{2\alpha\phi - \phi^2}{\alpha^2}$$

and the probability density function is,

$$f_{\alpha_i}(\phi) = \frac{d}{d\phi} P\{\alpha_i \leq \phi\} = \frac{2}{\alpha} \left(1 - \frac{\phi}{\alpha}\right)$$

The expected local progress is:

$$E(\cos \alpha_i) = \int_0^\alpha \cos \phi \cdot f_{\alpha_i}(\phi) d\phi = \frac{2(1 - \cos \alpha)}{\alpha^2}$$
Lemma

The expected “hops” efficiency of the “min-two uniform targets” protocol in the $\alpha$-uniform case is $E(C_h) \sim \frac{\alpha^2}{2(1-\cos \alpha)}$, for $0 \leq \alpha \leq \frac{\pi}{2}$ and for large $h$.

We remark that

$$\lim_{\alpha \to 0} E(C_h) = \lim_{\alpha \to 0} \frac{2\alpha}{2 \sin \alpha} = 1$$

and

$$\lim_{\alpha \to \frac{\pi}{2}} E(C_h) = \frac{(\pi/2)^2}{2(1-0)} = \frac{\pi^2}{8} \sim 1.24$$

Lemma

The expected “hops” efficiency of the min two uniform targets protocol is

$$1 \leq E(C_h) \leq \frac{\pi^2}{8} \sim 1.24$$

for large $h$ and for $0 \leq \alpha \leq \frac{\pi}{2}$. 
Consider $p$ at distance $x$ from $\mathcal{W}$. We assume that when $p$ searches in the sector $S$ defined by $(-\alpha, \alpha)$ and $R$, a particle $p'$ is returned in the sector with some probability density $f(p')dA$.

**Definition:** (Horizontal progress) Let $\Delta x$ be the projection of the line segment $(p, p')$ on the line from $p$ vertical to $\mathcal{W}$. Assume each search returns such a $p'$, with independent and identical distribution $f()$.

**Definition:** (Significant progress) Let $m > 0$ be the least integer such that $\mathbb{P}\{\Delta x > \frac{R}{m}\} \geq p$, where $0 < p < 1$ is a given constant.
**Definition:** Let the stochastic process $P$ where with probability $p$ the horizontal progress is $\mathcal{R}/m$ and with probability $q = 1 - p$ it is 0.

**Lemma:**

Let $Q$ the actual process. Then $P_P\{h \leq h_0\} \leq P_Q\{h \leq h_0\}$ (stochastic dominance).

Now let $t = \left\lceil \frac{x}{\mathcal{R}/m} \right\rceil = \left\lceil \frac{mx}{\mathcal{R}} \right\rceil$. Consider the integer r.v. $H$ such that $P\{H = i\} = q^i(1 - q)$ for any $i \geq 0$. Then $H$ is geometrically distributed. Let $H_1, \ldots, H_t$ be $t$ random variables, i.i.d. according to $H$. Clearly then:

**Lemma**

$P_P\{\# \text{ of hops is } h\} = P\{H_1 + \cdots + H_t = h\}$
Theorem
\[ P_P\{\text{the number of hops is } h\} = \binom{t}{h} p^t (-q)^h = \binom{t+h-1}{h} p^t q^h \]

(since \( h \) is negative binomial because it is the number of failures until the \( t_{th} \) success)

Corollary: For the process \( P \), the mean and variance of the number of hops are:

\[ E(h) = \frac{tq}{p}, \quad Var(h) = \frac{tq}{p^2} \]

The method above finds a distribution that upper bounds the number of hops. Since for all \( f() \) it is \( h \geq \frac{x}{R} = h_{opt} \) we get

Theorem
The process \( P \) estimates the expected number of hops with a guaranteed ratio \( \frac{(m+1)(1-p)}{p} \) at most.

Example: When for \( p = 0.5 \) we have \( m = 2 \) and the efficiency ratio is 3, i.e. the overestimate is 3 times the optimal number of hops.
Summary evaluation of LTP

- local, simple, greedy protocol
- no global structure (set of paths) maintained
- good for dense networks
- performance drops in sparse / faulty networks
C1. Multipath Routing

• multiple routes are used, towards increasing robustness
• the routes can be *disjoint or partially disjoint*
• **“Braided multipath routing”:**
  - There is a primary path that is used for routing.
  - Several *alternate paths* are maintained for use in case of a failure in the primary path.
  - **“Braided”** path: node disjointedness between alternate paths is not a strict requirement. Rather, for each node on the primary path, the method requires the existence of an alternate path from the source to the sink that *does not contain that node*, but which may otherwise overlap with the other nodes on the main path.
  - Compared to disjoint paths, the brained path is more suitable for *isolated* failures, while disjoint paths are more appropriate for *pattern (geographically correlated) failures*. 
Gradient Cost Routing (GRAd)

- All nodes in the network maintain an estimated cost to the sink (such as the number of hops needed to reach it).
- When a packet is transmitted, it includes a field with the cost “paid” already in the current data propagation. Also, a remaining value field is maintained, acting as a TTL (time-to-live) field for that packet.
- Any receiver that receives this packet forwards the packet iff its own cost to the sink is smaller than the remaining value of the packet. Before forwarding, the “cost paid” field is increased by one and the remaining value field is decreased by one.
- GRAd actually allows multiple nodes to forward the same message, so it essentially performs a limited directed flooding towards the sink and provides significant robustness but at the cost of a larger overhead.
Gradient Broadcast Routing (GRAB)

- It enhances GRAd by *incorporating a tunable energy-robustness trade-off through the use of credits.*
- Similarly to GRAd, GRAB maintains a cost field at each node. Additionally, the packets travel from a source to the sink with a *credit value* that is decreased at each step depending on the hop cost.
- **Credit-sharing mechanism:** *Earlier hops receive a larger share of the total credit in a packet,* while the later nodes receive a smaller share. Intermediate nodes with greater credit can spend a larger budget sending the packet to a larger set of forwarding neighbors. This way, paths *spread out initially* while eventually the *diverse paths converge to the sink* efficiently.
The GRAB Algorithm - Details (I)

- Each packet contains 3 fields:
  1. $R_o$: the credit assigned at the source node
  2. $C_o$: the cost-to-sink at the source node
  3. $U$: the budget already consumed from the source to the current hop

  (note: $R_o$, $C_o$ never change in the packet, while $U$ is increased in each hop).

- Each receiver $i$ (with a cost-to-sink $C_i$) computes the following metric:

  $$\beta = 1 - \frac{R_{oi}}{R_o}$$

  and a threshold $\theta$:

  $$\theta = \left(\frac{C_i}{C_o}\right)^2$$

  where $R_{oi} = U - (C_O - C_i)$

- Note: $C_o - C_i$ actually represents the optimal cost for travelling between source and node $i$, and $R_{oi}$ determines how much credit has been already used. Nodes far away from the optimal direct source-sink line will have a higher $U$ and thus a higher $R_{oi}$ credit spent. Thus, $\beta$ actually estimates how much credit is left for the packet.
• The packet is forwarded iff $\beta > \theta$, i.e. when the credit left is sufficient to cover the remaining cost (captured by $\theta$). The square exponent introduces a heavier weighting to $\theta$, in the sense that the threshold is more likely to be exceeded in the early hops than in the later hops, as desired (for limiting the path spreading). An example:

![Diagram of node 1 and node 2 with source and sink]

It is $U_2 > U_1$ and $C_2 > C_1 \Rightarrow R_{o2} > R_{o1} \Rightarrow \beta_2 < \beta_1$ and $\theta_2 > \theta_1 \Rightarrow$ the node 1 is more likely to forward than 2.

• The choice of the initial credit $R_o$ actually provides a tunable parameter to increase robustness at the expense of energy consumption, since broader paths are created (since $R_o \uparrow \Rightarrow \beta \uparrow$).
We assume that each node (particle) has the following abilities:

i) It can estimate the direction of a received transmission (e.g. via the technology of direction-sensing antennae).

ii) It can estimate the distance from a nearby particle that did the transmission (e.g. via estimation of the attenuation of the received signal).

iii) It knows the direction towards the sink S. This can be implemented during a set-up phase, where the sink broadcasts information about itself to all particles.

iv) All particles have a common co-ordinates system.

Note that GPS information is not needed for this protocol. Also, there is no need to know the global structure of the network.
• **Correctness.** Protocol $\Pi$ must guarantee that data arrives to the sink $S$, given that the network is operational.

• **Robustness.** $\Pi$ must guarantee that data arrives at enough points in a small interval around $S$, in cases when part of the network has become inoperative.

• **Efficiency.** $\Pi$ should have a small ratio of the number $k$ of activated particles over the total number $N$ of particles $r = \frac{k}{N}$. Thus $r$ is an energy efficiency measure of $\Pi$. 
PFR *probabilistically favors* redundant transmissions toward the sink within a *thin zone* of particles around the line connecting the particle sensing the event $\mathcal{E}$ and the sink.
Data is propagated with a suitably chosen probability $p$, while it is not propagated with probability $1 - p$.

To favor near-optimal transmissions the following probability is used:

$$P_{fwd} = \frac{\phi}{\pi}$$
The two phases of PFR

Phase 1: The “Front” Creation Phase. Initially (by using a limited, in terms of rounds, flooding) a sufficiently large “front" of particles is built, to guarantee the survivability of the data propagation process. Each particle having received the data, deterministically broadcasts it toward the sink.

Phase 2: The Probabilistic Forwarding Phase. Each particle $p$ receiving info($\epsilon$), broadcasts it to all its neighbors with probability $P_{fwd}$ (or it does not propagate any data with probability $1 - P_{fwd}$) defined as follows:

$$P_{fwd} = \begin{cases} 1 & \text{if } \phi \geq \phi_{threshold} = 134^\circ \\ \frac{\phi}{\pi} & \text{otherwise} \end{cases}$$
Lemma

PFR always succeeds in sending the information from $E$ to $S$ when the whole network is operational.

In the proof, geometry is used (i.e. we cover the network area by unit squares and show that there are always particles “close enough” to the optimal line, i.e. with $\phi > 134^\circ$, that deterministically broadcast).
The Energy Efficiency of PFR

- We consider particles that are active but *as far as possible* from ES

![Diagram of L_Q Area]

The particles inside the $L_Q$ Area

- We *approximate* $w$ by the following random walk

![Diagram of random walk]

$E$ $n_0$ $S$
By using stochastic dominance by a continuous time “discouraged arrivals” birth-death process, we prove:

**Theorem**

The energy efficiency of the PFR protocol is $\Theta\left(\left(\frac{n_0}{n}\right)^2\right)$ where $n_0 = |ES|$ and the total number of particles in the network is $N = n^2$. For $n_0 = o(n)$, this is $o(1)$. 
Particles *very near the ES line* are considered.

We study the case when *some* of these particles (at angles $> 134^\circ$) are *not operating*.

The probability that none of them transmits is *very small*.

It is shown:

**Lemma**

PFR manages to propagate the crucial data across lines parallel to $ES$, and of constant distance, with *fixed* nonzero probability.
The simulation environment:

- C++
- 2D geometry data types of LEDA
- a variety of sensor fields in a $100m \times 100m$ square area:
  - we drop randomly $n \in [100, 3000]$ particles (i.e. $0.01 \leq d \leq 0.3$)
  - fixed radio range $R = 5m$
  - search angle $\alpha = 90^\circ$
- we repeated each experiment more than 5000 times to get good average results
LTP – The impact of angle \( \alpha \)

- \( \alpha \to 0 \Rightarrow C_h \to 1 \)
  (assuming particles always exist)
- \( C_h \) initially decreases very fast, while having a limiting behavior for \( \alpha \leq 40 \)

Ideal Hops Efficiency for angles \( \alpha \in [5, 90] \)
LTP – The impact of sampling several targets

- $\#\text{ targets} \uparrow \Rightarrow C_h \to 1$
- 4 targets $\Rightarrow$ already very close to optimal
LTP – Failure rate

• for $d \leq 0.1$ both protocols almost always backtrack
• for $d \geq 0.2$ the failure rate drops very fast to 0.

![Graph showing failure rate for density $d \in [0.01, 0.5]$ and $\alpha = 90$.]
• PFR behaves very well ($r \leq 0.3$) for low densities ($d \leq 0.07$)
• PFR’s energy dissipation increases with density
• LTP performs best in dense networks
• For very low densities (i.e. $d \leq 0.12$), LTP backtracks a lot.
• As density increases, the number of backtracks of LTP reduces fast and almost reaches zero.
all protocols are near optimal (40 hops) even for low densities ($\geq 0.17$). PFR achieves this even for very low densities ($\geq 0.07$).

- LTP shows a pathological behavior for low densities ($\leq 0.12$) due to many backtracks.