Analytical Tools for the Performance Evaluation of Wireless Communication Systems

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Communication & Coding Theory for Wireless Channels

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Outline - Part I: Some Basics

1. Introduction: Background, Motivation, and Goals.
2. Fading Channels Characterization and Modeling (Brief Overview)
   - Multipath Fading
   - Shadowing
3. Single Channel Reception
   - Outage Probability
   - Average Fade/Outage Duration (AFD or AOD)
   - Average Probability of Error or Average Error Rate
     - Coherent Detection
     - Differentially Coherent and Noncoherent Detection

Design of Wireless Comm. Systems

- Often the basic problem facing the wireless system designer is to determine the "best" scheme in the face of his or her available constraints.
- An informed decision/choice relies on an accurate quantitative performance evaluation and comparison of various options and techniques.
- Performance of wireless communication systems can be measured in terms of:
  - Outage probability.
  - Average outage/fade duration.
  - Average bit or symbol error rate.

Performance Analysis

- Can lead to closed-form expressions or tractable solutions
  - Insight into performance limits and performance dependence on system parameters of interest.
  - A significant speed-up factor relative to computer simulations or field tests/experiments.
  - Quantify the tradeoff between performance and complexity.
  - Useful background study for accurate system design, improvement, and optimization.
- Approach
  - Mathematical and statistical modeling.
  - Analytical derivations.
  - Exact or approximate expressions in computable forms.
  - Numerical examples and design guidelines.

Fading Channels Characterization

- Wireless communications are subject to a complex and harsh radio propagation environment (multipath and shadowing).
- Considerable efforts have been devoted to the statistical modeling and characterization of these different effects resulting in a range of models for fading channels which depend on the particular propagation environment and the underlying communication scenario.
- Main characteristics of fading channels
  - Slow and fast fading channels.
  - Frequency-flat and frequency-selective fading channels.
- Characterization of slow and fast fading channels
  - Related to the **coherence time**, T_c, which measures the period of time over which the fading process is correlated

$$T_c \simeq \frac{1}{f_D}; \quad f_D : \text{Doppler spread.}$$

  - The fading is **slow** if the symbol time $T_s < T_c$ (i.e., fading constant over several symbols).
  - The fading is **fast** if the symbol time $T_s > T_c$.
  - In this lecture we focus on the performance of digital communication techniques over slow fading channels.

- Characterization of frequency-flat and frequency-selective channels.
  - Related to the **multipath intensity profile (MIP) or power delay profile (PDP)** $\phi_c(\tau)$.
  - **Delay spread or multipath spread** T_m is the maximum value of τ beyond which $\phi_c(\tau) \simeq 0$.
  - **Coherence bandwidth** is defined as

$$\Delta f_c \simeq \frac{1}{T_m}$$

  - **Frequency-flat or Frequency non-selective fading**
    * Signal components with frequency separation $(\Delta f) << (\Delta f)_c$ are completely correlated (affected in the same way by channel).
    * Typical of **narrowband** signals.
    * Since multipath delays are small compared to transmission baud interval, signal is not distorted (only attenuated) by the channel.
  - **Nonflat fading or Frequency-selective fading**
    * Signal components with frequency separation $(\Delta f) >> (\Delta f)_c$ are weakly correlated (affected differently by channel).
    * Typical of **wideband** signals (e.g. spread-spectrum signals).
    * Since multipath delays are large compared to transmission baud interval, signal is severely distorted (not only attenuated) by the channel.

Modeling of Frequency-Flat Fading Channels

- The received carrier amplitude is modulated by the random fading amplitude α
  - $\Omega = \overline{\alpha^2}$: average fading power of α.
  - $p_\alpha(\alpha)$: probability density function (PDF) of α.
- Let us denote the instantaneous signal-to-noise power ratio (SNR) per symbol by $\gamma = \alpha^2 E_s/N_0$ and the average SNR per symbol by $\overline{\gamma} = \Omega E_s/N_0$, where E_s is the energy per symbol.
- A standard transformation of the PDF $p_\alpha(\alpha)$ yields

$$p_\gamma(\gamma) = \frac{p_\alpha\left(\sqrt{\frac{\Omega\ \gamma}{\overline{\gamma}}}\right)}{2\sqrt{\frac{\gamma\overline{\gamma}}{\Omega}}}.$$

- Various statistical models
  - Multipath fading models
    * Rayleigh.
    * Nakagami-q (Hoyt).
    * Nakagami-n (Rice).
    * Nakagami-m.
  - Shadowing model
    * Log-normal.
  - Composite multipath/shadowing models.
    * Composite Nakagami-m/Log-normal.

Multipath Fading

- Rayleigh model
  - PDF of fading amplitude given by

$$p_\alpha(\alpha;\Omega) = \frac{2\ \alpha}{\Omega} \exp\left(-\frac{\alpha^2}{\Omega}\right); \quad \alpha \geq 0,$$

  - The instantaneous SNR per symbol of the channel, γ, is distributed according to an exponential distribution

$$p_\gamma(\gamma;\overline{\gamma}) = \frac{1}{\overline{\gamma}} \exp\left(-\frac{\gamma}{\overline{\gamma}}\right); \quad \gamma \geq 0.$$

  - Agrees very well with experimental data for multipath propagation where no line-of-sight (LOS) path exists between the transmitter and receiver antennas.
  - Applies to macrocellular radio mobile systems as well as to tropospheric, ionospheric, and maritime ship-to-ship communication.

- Nakagami-q (Hoyt) model
  - PDF of fading amplitude given by

$$p_\alpha(\alpha;\Omega,q) = \frac{(1+q^2)\ \alpha}{q\ \Omega} \exp\left(-\frac{(1+q^2)^2\ \alpha^2}{4q^2\Omega}\right) I_0\left(\frac{(1-q^4)\alpha^2}{4\ q^2\Omega}\right); \quad \alpha \geq 0,$$

    where $I_0(.)$ is the zero-th order modified Bessel function of the first kind, and q is the Nakagami-q fading parameter which ranges from 0 (half-Gaussian model) to 1 (Rayleigh model).

- – The instantaneous SNR per symbol of the channel, γ, is distributed according to

$$p_\gamma(\gamma;\overline{\gamma},q) = \frac{(1+q^2)}{2\;q\;\overline{\gamma}}\exp\left(-\frac{(1+q^2)^2\;\gamma}{4q^2\overline{\gamma}}\right)\;I_0\left(\frac{(1-q^4)\;\gamma}{4\;q^2\;\overline{\gamma}}\right);\quad \gamma \geq 0.$$

  – Applies to satellite links subject to strong ionospheric scintillation.

- Nakagami-n (Rice)
  – PDF of fading amplitude given by

$$p_\alpha(\alpha;\Omega,n) = \frac{2(1+n^2)e^{-n^2}\alpha}{\Omega}\exp\left(-\frac{(1+n^2)\alpha^2}{\Omega}\right)\;I_0\left(2n\alpha\sqrt{\frac{1+n^2}{\Omega}}\right);\;\alpha \geq 0,$$

  where n is the Nakagami-n fading parameter which ranges from 0 (Rayleigh model) to ∞ (AWGN channel) and which is related to the Rician K factor by $K = n^2$.

  – The instantaneous SNR per symbol of the channel, γ, is distributed according to

$$p_\gamma(\gamma;\overline{\gamma},n) = \frac{(1+n^2)e^{-n^2}}{\overline{\gamma}}\exp\left(-\frac{(1+n^2)\gamma}{\overline{\gamma}}\right)\;I_0\left(2n\sqrt{\frac{(1+n^2)\gamma}{\overline{\gamma}}}\right);\;\gamma \geq 0.$$

  – Applies to LOS paths of microcellular urban and suburban land mobile, picocellular indoor, and factory environments as well as to the dominant LOS path of satellite radio links.

- Nakagami-m
  - PDF of fading amplitude given by

$$p_\alpha(\alpha; \Omega, m) = \frac{2\, m^m\, \alpha^{2m-1}}{\Omega^m\, \Gamma(m)}\, \exp\left(-\frac{m\, \alpha^2}{\Omega}\right); \quad \alpha \geq 0,$$

    where m is the Nakagami-m fading parameter which ranges from 1/2 (half-Gaussian model) to ∞ (AWGN channel).
  - The instantaneous SNR per symbol of the channel, γ, is distributed according to a gamma distribution:

$$p_\gamma(\gamma; \overline{\gamma}, m) = \frac{m^m\, \gamma^{m-1}}{\overline{\gamma}^m\, \Gamma(m)}\, \exp\left(-\frac{m\, \gamma}{\overline{\gamma}}\right); \quad \gamma \geq 0.$$

  - Closely approximate the Nakagami-q (Hoyt) and the Nakagami-n (Rice) models.
  - Often gives the best fit to land-mobile and indoor-mobile multipath propagation, as well as scintillating ionospheric radio links.

Nakagami-m PDF

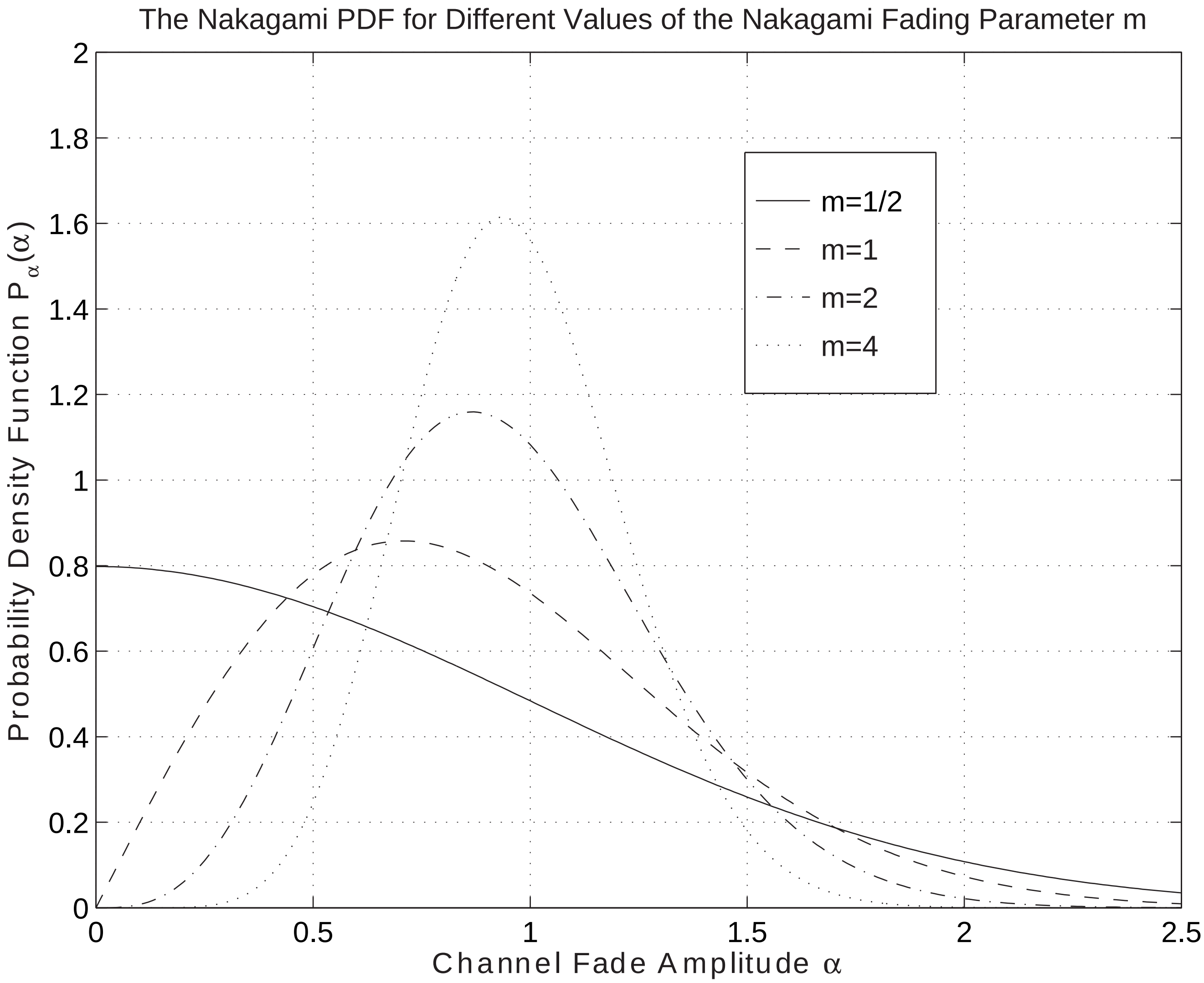


Figure 1: The Nakagami PDF for different values of the Nakagami fading parameter m.

Shadowing and Composite Effect

- **Log-normal shadowing**
  - Due to the shadowing of the received signal by obstructions such as building, trees, and hills.
  - Empirical measurements support a log-normal distribution:

$$p_\sigma(\gamma;\mu,\sigma) = \frac{\xi}{\sqrt{2\pi}\ \sigma\ \gamma}\ \exp\left[-\frac{(10\log_{10}\gamma-\mu)^2}{2\ \sigma^2}\right],$$

    where $\xi = 10/\ln 10 = 4.3429$, and μ (dB) and σ (dB) are the mean and the standard deviation of $10\log_{10}\gamma$, respectively.

- **Composite Multipath/Shadowing**
  - Consists of multipath fading superimposed on log-normal shadowing.
  - Example: composite Nakagami-m/log-normal PDF [Ho and Stüber]

$$\begin{aligned} p_\gamma(\gamma;m,\mu,\sigma) &= \int_0^\infty \frac{m^m\ \gamma^{m-1}}{w^m\ \Gamma(m)}\ \exp\left[-\frac{m\ \gamma}{w}\right] \\ &\times\ \frac{\xi}{\sqrt{2\pi}\ \sigma\ w}\ \exp\left[-\frac{(10\log_{10}w-\mu)^2}{2\ \sigma^2}\right]\ dw. \end{aligned}$$

  - Often the scenario in congested downtown areas with slow moving pedestrians and vehicles. This type of composite fading is also observed in land-mobile satellite systems subject to vegetative and/or urban shadowing.

Modeling of Frequency-Selective Fading Channels

- Frequency-selective fading channels can be modeled by a linear filter characterized by the following complex-valued lowpass equivalent impulse response

$$h(t) = \sum_{l=1}^{L_p} \alpha_l \, e^{-j\theta_l} \, \delta(t - \tau_l),$$

  where

  - $\delta(.)$ is the Dirac delta function.
  - l is the path index.
  - L_p is the number of propagation paths and is related to the ratio of the delay spread to the symbol time duration.
  - $\{\alpha_l\}_{l=1}^{L_p}$, $\{\theta_l\}_{l=1}^{L_p}$, and $\{\tau_l\}_{l=1}^{L_p}$ are the random channel amplitudes, phases, and delays, respectively.

- The fading amplitude α_l of the lth "resolvable" path is assumed to be a random variable with average fading power $\overline{\alpha_l^2}$ denoted by Ω_l and with PDF $p_{\alpha_l}(\alpha_l)$ which can follow any one of the models presented above.

- The $\{\Omega_l\}_{l=1}^{L_p}$ are related to the channel's power delay profile or multipath intensity profile and which is typically a decreasing function of the delay. Example: exponentially decaying profile for indoor office buildings and congested urban areas:

$$\Omega_l = \Omega_1 \, e^{-\tau_l/T_m}; \quad l = 1, 2, \cdots, L_p,$$

  where Ω_1 is the average fading power corresponding to the first (reference) propagation path.

Outage Probability and Outage Duration

- Outage Probability
  - Usually defined as the probability that the instantaneous bit error rate (BER) exceeds a certain target BER.
  - Equivalently it is the probability that the instantaneous SNR γ falls below a certain target SNR $\gamma_{\rm th}$:

$$P_{\rm out}(\gamma_{\rm th}) = {\rm Prob}[\gamma \leq \gamma_{\rm th}] = \int_0^{\gamma_{\rm th}} p_\gamma(\gamma)\, d\gamma = P_\gamma(\gamma_{\rm th}),$$

    where $P_\gamma(\cdot)$ is the SNR cumulative distribution function (CDF).

- Average Outage Duration
  - Usually defined as the average time that the instantaneous BER remains above a certain target BER once it exceeds it.
  - Equivalently it is the average time that the instantaneous SNR $\gamma(t)$ remains below a certain target SNR $\gamma_{\rm th}$ once it drops below it:

$$T(\gamma_{\rm th}) = \frac{P_{\rm out}(\gamma_{\rm th})}{N(\gamma_{\rm th})},$$

    where $N(\gamma_{\rm th})$ is the average (up-ward or down-ward) crossing rate of $\gamma(t)$ at level $\gamma_{\rm th}$.

Examples for CDFs of the SNR

- Rayleigh fading

$$P_\gamma(\gamma) = 1 - e^{-\frac{\gamma}{\overline{\gamma}}}.$$

- Nakagami-n (Rice with $K = n^2$) fading

$$P_\gamma(\gamma) = 1 - Q\left(n\sqrt{2}, \sqrt{\frac{2(1+n^2)}{\overline{\gamma}}\gamma}\right),$$

where $Q(\cdot,\cdot)$ is the Marcum Q-function traditionally defined by

$$Q(a,b) = \int_b^\infty x \exp\left(-\frac{x^2+a^2}{2}\right) I_0(ax)\, dx.$$

- Nakagami-m fading

$$P_\gamma(\gamma) = 1 - \frac{\Gamma\left(m, \frac{m}{\overline{\gamma}}\gamma\right)}{\Gamma(m)},$$

where $\Gamma(\cdot,\cdot)$ is the complementary incomplete gamma function traditionally defined by

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t}\, t^{\alpha-1}\, dt.$$

Average Bit Error Rate

- Average bit or symbol error rate.
  - Coherent Detection
    * Conditional (on the instantaneous SNR) BER for BPSK for example

$$P_b(E/\gamma) = Q\left(\sqrt{2\gamma}\right),$$

      where $Q(\cdot)$ is the Gaussian Q-function traditionally defined by

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt.$$

    * Average BER

$$P_b(E) = \int_0^\infty P_b(E/\gamma)\, p_\gamma(\gamma)\, d\gamma.$$

  - Differentially Coherent and Noncoherent Detection
    * Conditional (on the instantaneous SNR) BER for DPSK for example

$$P_b(E/\gamma) = \frac{1}{2} e^{-\gamma},$$

    * Average BER of DPSK

$$P_b(E) = \int_0^\infty P_b(E/\gamma)\, p_\gamma(\gamma)\, d\gamma = \frac{1}{2}\mathcal{M}_\gamma(-1),$$

      where $\mathcal{M}_\gamma(s) = E_\gamma[e^{s\gamma}]$ is the moment generating function (MGF) of the SNR.

Examples for MGFs of the SNR

- Rayleigh fading

$$\mathcal{M}_\gamma(s) = (1 - s\overline{\gamma})^{-1}.$$

- Nakagami-n (Rice with $K = n^2$) fading

$$\mathcal{M}_\gamma(s) = \frac{1+n^2}{1+n^2-s\overline{\gamma}} \exp\left(\frac{s\overline{\gamma}n^2}{1+n^2-s\overline{\gamma}}\right).$$

- Nakagami-m fading

$$\mathcal{M}_\gamma(s) = \left(1 - \frac{s\overline{\gamma}}{m}\right)^{-m}.$$

- Composite Nakagami-m/log-normal fading

$$\mathcal{M}_\gamma(s) \simeq \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N_p} H_{x_n} \left(1 - s\frac{10^{(\sqrt{2}\ \sigma\ x_n+\mu)/10}}{m}\right)^{-m},$$

  where

  - N_p is the order of the Hermite polynomial, $H_{N_p}(.)$. Setting N_p to 20 is typically sufficient for excellent accuracy.
  - x_n are the zeros of the N_p-order Hermite polynomial.
  - H_{x_n} are the weight factors of the N_p-order Hermite polynomial.

Outline - Part II: Diversity Systems

1. Introduction: Concept, Intuition, and Notations.
2. Classification of Diversity Combining Techniques
3. Receiver Diversity Techniques
   - "Pure" Diversity Combining Techniques
     - Maximal-Ratio Combining (MRC)
     - Equal Gain Combining (EGC)
     - Selection Combining (SC)
     - Switched and Stay Combining (SSC) and Switch-and-Examine Combining (SEC)
   - "Hybrid" Diversity Combining Techniques
     - Generalized Selection Combining (GSC) and Generalized Switch-and-Examine Combining (SEC)
     - Two-Dimensional Diversity Schemes
4. Impact of Correlation on the Performance of Diversity Systems
5. Transmit Diversity Systems
6. Multiple-Input-Multiple-Output (MIMO) Systems

Diversity Combining

- **Concept**
  - Diversity combining consists of:
    * Receiving redundantly the same information bearing signal over 2 or more fading channels.
    * Combining these multiple replicas at the receiver in order to increase the overall received SNR.
- **Intuition**
  - The intuition behind diversity combining is to take advantage of the low probability of concurrence of deep fades in all the diversity branches to lower the probability of error and outage.
- **Means of Realizing Diversity**
  - Multiple replicas can be obtained by extracting the signals via different radio paths:
    * Space: Multiple receiver antennas (antenna or site diversity).
    * Frequency: Multiple frequency channels which are separated by at least the coherence bandwidth of the channel (frequency hopping or multicarrier systems).
    * Time: Multiple time slots which are separated by at least the coherence time of the channel (coded systems).
    * Multipath: Resolving multipath components at different delays (direct-sequence spread-spectrum systems with RAKE reception).

Multilink Channel Model

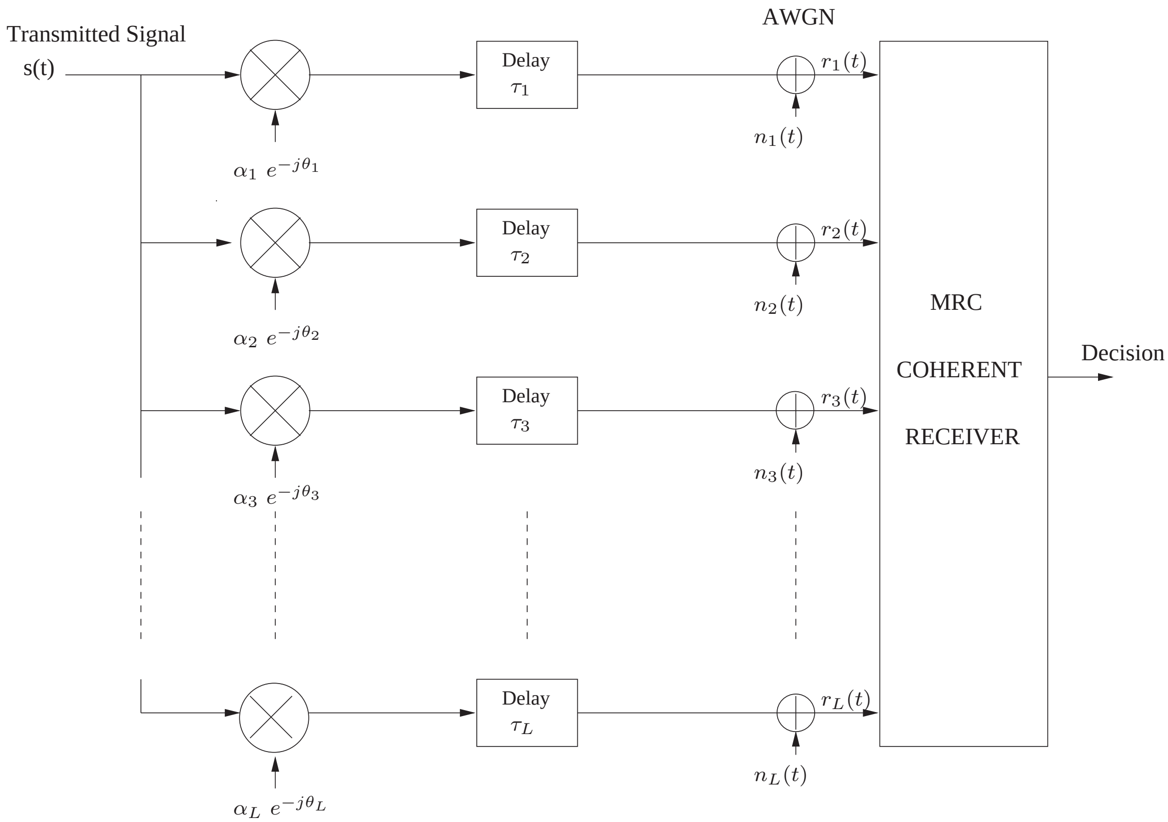


Figure 2: Multilink channel model.

Notations and Assumptions for Multichannel Reception

- Transmitted complex signal denoted by $\tilde{s}(t)$ (corresponding to any of the modulation types).
- Multilink channel model: Transmitted signal is received over L separate channels resulting in the set of received replicas $\{r_l(t)\}_{l=1}^{L}$ characterized by the sets $\{\alpha_l\}_{l=1}^{L}$, $\{\theta_l\}_{l=1}^{L}$, and $\{\tau_l\}_{l=1}^{L}$.
- Received complex signal is denoted by $\tilde{r}_l(t)$:

$$\tilde{r}_l(t) = \alpha_l \, e^{-j\theta_l} \, \tilde{s}(t - \tau_l) + \tilde{n}_l(t), \;\; l = 1, 2, \cdots, L.$$

  - α_l: fading amplitude of the lth path with PDF denoted by $p_{\alpha_l}(\alpha_l)$. Examples of $p_{\alpha_l}(\alpha_l)$ are Rayleigh, Nakagami-n (Rice), or Nakagami-m.
  - θ_l: fading phase of the lth path.
  - τ_l: fading delay of lth path.
  - $\tilde{n}_l(t)$: additive white Gaussian noise (AWGN).
- Independence assumption: the sets $\{\alpha_l\}_{l=1}^{L}$, $\{\theta_l\}_{l=1}^{L}$, $\{\tau_l\}_{l=1}^{L}$, and $\{\tilde{n}_l(t)\}_{l=1}^{L}$ are assumed to be independent of one another.
- Slow fading assumption: the sets $\{\alpha_l\}_{l=1}^{L}$, $\{\theta_l\}_{l=1}^{L}$, and $\{\tau_l\}_{l=1}^{L}$ are assumed to be constant over at least one symbol time.
- Two convenient parameters:
  - $\gamma_l = \alpha_l^2 \; E_s/N_{0_l}$: instantaneous SNR per symbol of lth path.
  - $\overline{\gamma}_l = E_{\alpha_l}\left[\alpha_l^2\right] E_s/N_{0_l} = \Omega_l E_s/N_{0_l}$: average SNR per symbol of the lth path.

Classification of Diversity Systems

- Macroscopic versus microscopic diversity:
  1. Macroscopic diversity mitigates the effect of shadowing.
  2. Microscopic diversity mitigates the effect of multipath fading.
- "Soft" versus "Hard" diversity schemes:
  1. Soft diversity combining schemes deal with signals.
  2. Hard diversity combining schemes deal with bits.
- Receive versus transmit diversity schemes:
  1. In receive diversity systems, the diversity is extracted at the receiver (for example multiple antennas deployed at the receiver).
  2. In transmit diversity systems, the diversity is initiated at the transmitter (for example multiple antennas deployed at the transmitter).
  3. MIMO systems, such as systems with multiple antennas at the transmitter and the receiver, take advantage of diversity at both the receiver and transmitter ends.
- Pre-detection versus post-detection combining:
  1. Pre-detection combining: diversity combining takes place before detection.
  2. Post-detection combining: diversity combining takes place after detection.

Diversity Combining Techniques

- Four "pure" types of diversity combining techniques:
  - Maximal-ratio combining (MRC)
    * Optimal scheme but requires knowledge of all channel parameters (i.e., fading amplitude and phase of every diversity path).
    * Used with coherent modulations.
  - Equal gain combining (EGC)
    * Coherent version limited in practice to constant envelope modulations.
    * Noncoherent version optimum in the maximum-likelihood sense for i.i.d. Rayleigh channels.
  - Selection combining (SC)
    * Uses the diversity path/branch with the best quality.
    * Requires simultaneous and continuous monitoring of all diversity branches.
  - Switched (or scanning diversity)
    * Two variants: Switch-and-stay combining (SSC) and switch-and-examine combining (SEC).
    * Least complex diversity scheme.
- "Hybrid" diversity schemes
  - Generalized selection combining (GSC) and generalized switch-and-examine combining (GSEC)
  - Two-dimensional diversity schemes.

Designer Problem

- Once the modulation scheme and the means of creating multiple replicas of the same signal are chosen, the basic problem facing the wireless system designer becomes one of determining the "best" diversity combining scheme in the face of his or her available constraints.
- An informed decision/choice relies on an accurate quantitative performance evaluation of these various combining techniques when used in conjunction with the chosen modulation.
- Performance of diversity systems can be measured in terms of:
  - Average SNR after combining.
  - Outage probability of the combined SNR
  - Average outage duration.
  - Average bit or symbol error rate.
- Performance of diversity systems is affected by various channel parameters such as:
  - **Fading distribution on the different diversity paths**. For example for multipath diversity the statistics of the different paths may be statistically characterized by different families of distributions.
  - **Average fading power**. For example in multipath diversity the average fading power is typically assumed to follow an exponentially decaying power delay profile: $\overline{\gamma}_l = \overline{\gamma}_1 \; e^{-\delta \; (l-1)}$ $(l = 1, 2, \cdots, L_p)$, where δ is average fading power decay factor.

- – **Severity of fading**. For example fading in macrocellular environment tends to follow Rayleigh type of fading while fading tends to be Rician or Nakagami-m in microcellular type of environment.
  – **Fading correlation**. For example because of insufficient antenna spacing in small-size mobile units equipped with space antenna diversity. In this case the maximum theoretical diversity gain cannot be achieved.

Objective

- Develop "generic" analytical tools to assess the performance of diversity combining techniques in various wireless fading environments.

Maximal-Ratio Combining (MRC)

- Let L_c denote the number of combined channels, E_b the energy-per-bit, α_l the fading amplitude of the lth channel, and N_l the noise spectral density of the lth channel.
- For MRC the conditional (on fading amplitudes $\{\alpha_l\}_{l=1}^{L_c}$) combined SNR per bit, γ_t is given by

$$\gamma_t = \sum_{l=1}^{L_c} \frac{E_b}{N_l}\alpha_l^2 = \sum_{l=1}^{L_c} \gamma_l.$$

- For binary coherent signals the conditional error probability is

$$P_b\left(E|\{\gamma_l\}_{l=1}^{L_c}\right) = Q\left(\sqrt{\sum_{l=1}^{L_c} 2g\frac{E_b}{N_l}\alpha_l^2}\right) = Q\left(\sqrt{2g\sum_{l=1}^{L_c}\gamma_l}\right),$$

  $g = 1$ for BPSK, $g = 1/2$ for orthogonal BFSK, and $g = 0.715$ for BFSK with minimum correlation.
- Average error probability is

$$P_b(E) = \underbrace{\int_0^\infty \cdots \int_0^\infty}_{L_c-\text{fold}} Q\left(\sqrt{2g\sum_{l=1}^{L_c}\gamma_l}\right) p_{\gamma_1,\gamma_2,\cdots,\gamma_{L_c}}(\gamma_1,\cdots,\gamma_{L_c})\, d\gamma_1\cdots d\gamma_{L_c},$$

  where $p_{\gamma_1,\gamma_2,\cdots,\gamma_{L_c}}(\gamma_1,\gamma_2,\cdots,\gamma_{L_c})$ is the joint PDF of the $\{\gamma_l\}_{l=1}^{L_c}$.
- Two approaches to simplify this L_c-fold integral:
  - Classical PDF-based approach.
  - MGF-based approach which relies on the alternate representation of the Gaussian Q-function.

PDF-Based Approach

- Find the distribution of $\gamma_t = \sum_{l=1}^{L_c} \gamma_l$, $p_{\gamma_t}(\gamma_t)$, then replace the L_c-fold average by a single average over γ_t

$$P_b(E) = \int_0^\infty Q\left(\sqrt{2g\gamma_t}\right)\, p_{\gamma_t}(\gamma_t)\, d\gamma_t.$$

- Requires finding the distribution of γ_t in a simple form.
- If this is possible, it can lead to a closed form expression for the average probability of error.
- **Example:** MRC combining of L_c independent identically distributed (i.i.d.) Rayleigh fading paths [Proakis Textbook]
  - The SNR per bit per path γ_l has an exponential distribution with average SNR per bit $\overline{\gamma}$

$$p_{\gamma_l}(\gamma_l) = \frac{1}{\overline{\gamma}} e^{-\gamma_l/\overline{\gamma}}.$$

  - The SNR per bit of the combined SNR $\gamma_t = \sum_{l=1}^{L_c} \gamma_l$ has a gamma distribution

$$p_{\gamma_t}(\gamma_t) = \frac{1}{(L_c-1)!\overline{\gamma}^{L_c}}\, \gamma_t^{L_c-1}\, e^{-\gamma_t/\overline{\gamma}}.$$

  - The average probability of error can be found in closed-form by successive integration by parts

$$P_b(E) = \left(\frac{1-\mu}{2}\right)^{L_c} \sum_{l=0}^{L_c-1} \binom{L_c-1+l}{l} \left(\frac{1+\mu}{2}\right)^{l},$$

where

$$\mu = \sqrt{\frac{\overline{\gamma}}{1+\overline{\gamma}}}.$$

Limitations of the PDF-Based Approach

- Finding the PDF of the combined SNR per bit γ_t in a simple form is typically feasible if the paths are i.i.d.
- More difficult problem if the combined paths are correlated or come from the same family of fading distribution (e.g., Rice) but have different parameters (e.g., different average fading powers (i.e., a nonuniform power delay profile) and/or different severity of fading parameters).
- Intractable in a simple form if the paths have fading distributions coming from different families of distributions or if they have an arbitrary correlation profile.
- We now show how the alternative representation of the Gaussian Q-function provides a simple and elegant solution to many of these limitations.

Alternative Form of the Gaussian Q-function

- The Gaussian Q-function is traditionally defined by

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2}\, dt.$$

  - The argument x is in the lower limit of the integral.

- A preferred representation of the Gaussian Q-function is given by [Nuttal 72, Weinstein 74, Pawula *et al.* 78, and Craig 91]

$$Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{x^2}{2\,\sin^2\phi}\right) d\phi; \quad x \geq 0.$$

  - Finite-range integration.
  - Limits are independent of the argument x.
  - Integrand is exponential in the argument x.

- Additional property of alternate representation
  - Integrand is maximum at $\phi = \pi/2$.
  - Replacing the integrand by its maximum value yields

$$Q(x) \leq \frac{1}{2}\, e^{-x^2/2}; \quad x \geq 0,$$

    which is the well-known Chernoff bound.

MGF-Based Approach

- Assuming independent (but not necessarily identically distributed) fading paths amplitudes

$$p_{\gamma_1,\gamma_2,\cdots,\gamma_{L_c}}(\gamma_1,\cdots,\gamma_{L_c}) = \prod_{l=1}^{L_c} p_{\gamma_l}(\gamma_l)$$

- Using alternate representation of the Gaussian Q-function:

$$P_b(E) = \frac{1}{\pi}\underbrace{\int_0^\infty \cdots \int_0^\infty}_{L_c-\text{fold}} \int_0^{\pi/2} \exp\left(-\frac{g\sum_{l=1}^{L_c}\gamma_l}{\sin^2\phi}\right) d\phi \prod_{l=1}^{L_c} p_{\gamma_l}(\gamma_l)\, d\gamma_1 \cdots d\gamma_{L_c}.$$

- Take advantage of the product form by writing the exponential of the sum as the product of exponentials

$$\exp\left(-\frac{g\sum_{l=1}^{L_c}\gamma_l}{\sin^2\phi}\right) = \prod_{l=1}^{L_c} \exp\left(-\frac{g\ \gamma_l}{\sin^2\phi}\right).$$

- Grouping like terms (i.e. terms of index l) and switching order of integration allows partitioning of the L_c-fold integral into a product of L_c one-dimensional integrals:

$$\begin{aligned} P_b(E) &= \frac{1}{\pi}\int_0^{\pi/2} \prod_{l=1}^{L_c} \underbrace{\int_0^\infty p_{\gamma_l}(\gamma_l) \exp\left(-\frac{g\gamma_l}{\sin^2\phi}\right) d\gamma_l}_{\mathcal{M}_{\gamma_l}\left(-\frac{g}{\sin^2\phi};\overline{\gamma}_l\right)} d\phi \\ &= \frac{1}{\pi}\int_0^{\pi/2} \prod_{l=1}^{L_c} \mathcal{M}_{\gamma_l}\left(-\frac{g}{\sin^2\phi};\overline{\gamma}_l\right) d\phi, \end{aligned}$$

where $\mathcal{M}_{\gamma_l}(s;\overline{\gamma}_l)$ denotes the MGF of the lth path with average SNR per bit $\overline{\gamma}_l$.

MGF-Based Approach - Examples

- Nakagami-q (Hoyt) fading

$$\mathcal{M}_{\gamma_l}\left(-\frac{g}{\sin^2\phi};\overline{\gamma}_l\right)=\left(1+\frac{2\,\overline{\gamma}_l}{\sin^2\phi}+\frac{4\,q_l^2\,\overline{\gamma}_l^2}{(1+q_l^2)^2\,\sin^4\phi}\right)^{-1/2}.$$

- Nakagami-n (Rice) fading

$$\mathcal{M}_{\gamma_l}\left(-\frac{g}{\sin^2\phi};\overline{\gamma}_l\right)=\frac{(1+n_l^2)\,\sin^2\phi}{(1+n_l^2)\sin^2\phi+\overline{\gamma}_l}\,\exp\left(-\frac{n_l^2\,\overline{\gamma}_l}{(1+n_l^2)\sin^2\phi+\overline{\gamma}_l}\right).$$

- Nakagami-m fading

$$\mathcal{M}_{\gamma_l}\left(-\frac{g}{\sin^2\phi};\overline{\gamma}_l\right)=\left(1+\frac{\overline{\gamma}_l}{m_l\,\sin^2\phi}\right)^{-m_l}.$$

- Composite Nakagami-m/log-normal fading

$$\mathcal{M}_{\gamma_l}\left(-\frac{g}{\sin^2\phi};\mu_l\right)\simeq\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N_p}H_{x_n}\left(1+\frac{10^{(\sqrt{2}\,\sigma_l\,x_n+\mu_l)/10}}{m_l\,\sin^2\phi}\right)^{-m_l},$$

  where

  - N_p is the order of the Hermite polynomial, $H_{N_p}(.)$. Setting N_p to 20 is typically sufficient for excellent accuracy.
  - x_n are the zeros of the N_p-order Hermite polynomial.
  - H_{x_n} are the weight factors of the N_p-order Hermite polynomial.

Advantages of MGF-Based Approach

- Alternate representation of the the Gaussian Q-function allows partitioning of the integrand so that the averaging over the fading amplitudes can be done independently for each path *regardless of whether the paths are identically distributed or not.*

- *Desired* representations of the conditional symbol error rate of M-PSK and M-QAM allows obtaining the average symbol error rate in a generic fashion with the MGF-based approach:

  – For M-PSK the average symbol error rate is given by

$$P_s(E) = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \prod_{l=1}^{L_c} \mathcal{M}_{\gamma_l}\left(-\frac{g_{\text{psk}}}{\sin^2\phi};\overline{\gamma}_l\right)\, d\phi,$$

  where $g_{\text{psk}} = \sin^2(\pi/M)$.

  – For M-QAM the average symbol error rate is given by

$$\begin{aligned} P_s(E) &= \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)\int_0^{\pi/2} \prod_{l=1}^{L_c} \mathcal{M}_{\gamma_l}\left(-\frac{g_{\text{qam}}}{\sin^2\phi};\overline{\gamma}_l\right)\, d\phi \\ &- \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)^2\int_0^{\pi/4} \prod_{l=1}^{L_c} \mathcal{M}_{\gamma_l}\left(-\frac{g_{\text{qam}}}{\sin^2\phi};\overline{\gamma}_l\right)\, d\phi, \end{aligned}$$

  where $g_{\text{qam}} = \frac{3}{2(M-1)}$.

- MGF-based approach can still provide an elegant and general solution for Nakagami-m correlated combined paths.

Switched Diversity

- **Motivation**
  - MRC and EGC require all or some of the channel state information (fading amplitude, phase, and delay) from all the received signals.
  - For MRC and EGC a separate receiver chain is needed for each diversity branch, which adds to the overall receiver complexity.
  - SC type systems only process one of the diversity branches but may be not very practical in its conventional form since it still requires the simultaneous and continuous monitoring of all the diversity branches.
  - SC often implemented in the form of switched diversity.
- **Mode of Operation**
  - Receiver selects a particular branch until its SNR drops below a predetermined threshold.
  - When this happens the receiver switches to another branch.
  - For dual branch switch and stay combining (SSC) the receiver switches to, and stays with, the other branch regardless of whether or not the SNR of that branch is above or below the predetermined threshold.

CDF and PDF of SSC Output

- Let $\gamma_{\rm ssc}$ denote the the SNR per bit at the output of the SSC combiner and let γ_T denote the predetermined switching threshold.
- The CDF of SSC output is defined by

$$P_{\gamma_{\rm ssc}}(\gamma) = \mathrm{Prob}[\gamma_{\rm ssc} \leq \gamma]$$

- Assuming that the two combined branches are i.i.d. then

$$P_{\gamma_{\rm ssc}}(\gamma) = \begin{cases} \mathrm{Prob}[(\gamma_1 \leq \gamma_T) \text{ and } (\gamma_2 \leq \gamma)], & \gamma < \gamma_T \\ \mathrm{Prob}[(\gamma_T \leq \gamma_1 \leq \gamma) \text{ or } (\gamma_1 \leq \gamma_T \text{ and } \gamma_2 \leq \gamma)] & \gamma \geq \gamma_T, \end{cases}$$

which can be expressed in terms of the CDF of the individual branches, $P_\gamma(\gamma)$, as

$$P_{\gamma_{\rm ssc}}(\gamma) = \begin{cases} P_\gamma(\gamma_T)\ P_\gamma(\gamma) & \gamma < \gamma_T \\ P_\gamma(\gamma) - P_\gamma(\gamma_T) + P_\gamma(\gamma)\ P_\gamma(\gamma_T) & \gamma \geq \gamma_T. \end{cases}$$

- Differentiating $P_{\gamma_{\rm ssc}}(\gamma)$ with respect to γ we get the PDF of the SSC output in terms of the CDF $P_\gamma(\gamma)$ and the PDF $p_\gamma(\gamma)$ of the individual branches

$$p_{\gamma_{\rm ssc}}(\gamma) = \frac{dP_{\gamma_{\rm ssc}}(\gamma)}{d\gamma} = \begin{cases} P_\gamma(\gamma_T)\ p_\gamma(\gamma) & \gamma < \gamma_T \\ (1 + P_\gamma(\gamma_T))\ p_\gamma(\gamma) & \gamma \geq \gamma_T. \end{cases}$$

- For example for Nakagami-m fading

$$p_\gamma(\gamma) = \frac{m^m\ \gamma^{m-1}}{\overline{\gamma}^m\ \Gamma(m)}\ \exp\left(-\frac{m\ \gamma}{\overline{\gamma}}\right); \quad \gamma \geq 0.$$

$$P_\gamma(\gamma) = 1 - \frac{\Gamma\left(m, \frac{m}{\overline{\gamma}}\gamma\right)}{\Gamma(m)}; \quad \gamma \geq 0.$$

.

Average BER of BPSK

- Let $P_b(E|\gamma)$ denote the conditional BER and $P_{b_o}(E;\overline{\gamma})$ denote the average BER with no diversity.
- Average BER with SSC is given by

$$\begin{aligned} P_b(E) &= \int_o^\infty P_b(E|\gamma)\; p_{\gamma_{\rm ssc}}(\gamma)\; d\gamma \\ &= \int_0^\infty P_b(E|\gamma)\; P_\gamma(\gamma_T)\; p_\gamma(\gamma)\; d\gamma + \int_{\gamma_T}^\infty P_b(E|\gamma)\; p_\gamma(\gamma)\; d\gamma. \end{aligned}$$

- Using alternate representation of the Gaussian $Q(\cdot)$ function in the conditional BER then switching the order of integration we get

$$\begin{aligned} P_b(E) &= \frac{1}{\pi}\int_0^{\pi/2}\left[\int_0^\infty e^{-\frac{\gamma}{\sin^2\phi}}\; P_\gamma(\gamma_T)\; p_\gamma(\gamma)\; d\gamma + \int_{\gamma_T}^\infty e^{-\frac{\gamma}{\sin^2\phi}}\; p_\gamma(\gamma)\; d\gamma\right] d\phi \\ &= \frac{1}{\pi}\int_0^{\pi/2} P_\gamma(\gamma_T)\; \mathcal{M}_\gamma\left(-\frac{1}{\sin^2\phi}\right) d\phi + \frac{1}{\pi}\int_0^{\pi/2}\left[\int_{\gamma_T}^\infty p_\gamma(\gamma) e^{-\frac{\gamma}{\sin^2\phi}} d\gamma\right] d\phi. \end{aligned}$$

- For Rayleigh, Nakagami-n (Rice), and Nakagami-m type of fading the integrand of the second integral can be expressed in closed-form in terms of tabulated functions. Hence the final result is in the form of a single finite-range integral.
- For example for Nakagami-m fading channels the final result involves the incomplete Gamma function $\Gamma(\cdot,\cdot)$:

$$\begin{aligned} P_b(E) = \frac{1}{\pi}\int_0^{\pi/2} & \left(1 + \frac{\overline{\gamma}}{m\sin^2\phi}\right)^{-m} \\ \times & \left(1 + \frac{\Gamma\left(m, \left(\frac{m}{\overline{\gamma}} + \frac{1}{\sin^2\phi}\right)\gamma_T\right) - \Gamma\left(m, \frac{m\gamma_T}{\overline{\gamma}}\right)}{\Gamma(m)}\right) d\phi. \end{aligned}$$

Optimum Threshold

- The setting of the predetermined threshold is an additional important system design issue for SSC diversity systems.
- If the threshold level is chosen too high, the switching unit is almost continually switching between the two antennas which results not only in a poor diversity gain but also in an undesirable increase in the rate of the switching transients on the transmitted data stream.
- If the threshold level is chosen too low, the switching unit is almost locked to one of the diversity branches, even when the SNR level is quite low, and again there is little diversity gain achieved.
- There exists an optimum threshold, in a minimum average error rate sense, which is denoted by γ_T^* and which is a solution of the equation

$$\frac{dP_b(E)}{d\gamma_T}\bigg|_{\gamma_T=\gamma_T^*} = 0.$$

- Differentiating the previously obtained expression for the average BER with respect to γ_T we get

$$\frac{1}{\pi}\int_0^{\pi/2} p_\gamma(\gamma_T^*)\mathcal{M}_\gamma\left(-\frac{1}{\sin^2\phi}\right)d\phi - \frac{1}{\pi}\int_0^{\pi/2} p_\gamma(\gamma_T^*)e^{-\frac{\gamma_T^*}{\sin^2\phi}}\,d\phi = 0,$$

which after simplification reduces to

$$P_{b_o}(E;\overline{\gamma}) - Q\left(\sqrt{2\gamma_T^*}\right) = 0.$$

- Solving for γ_T^* in the previous equation leads to the desired expression for the optimum threshold given by

$$\gamma_T^* = \frac{1}{2}\left[Q^{-1}\left(P_{b_o}(E;\overline{\gamma})\right)\right]^2,$$

where $Q^{-1}(\cdot)$ denotes the inverse Gaussian $Q(\cdot)$-function.

- For example:

  – For Rayleigh fading

$$\gamma_T^* = \frac{1}{2}\left[Q^{-1}\left(\frac{1}{2}\left(1-\sqrt{\frac{\overline{\gamma}}{1+\overline{\gamma}}}\right)\right)\right]^2.$$

  – For Nakagami-m fading

$$\gamma_T^* = \frac{1}{2}\left[Q^{-1}\left(\frac{\sqrt{\frac{\overline{\gamma}}{\pi m}}}{2\,(1+\frac{\overline{\gamma}}{m})^{m+1/2}}\,\frac{\Gamma(m+1/2)}{\Gamma(m+1)}\;{}_2F_1\left(1, m+\frac{1}{2}; m+1; \frac{1}{1+\frac{\overline{\gamma}}{m}}\right)\right)\right]^2.$$

- In summary:

  – Alternate representations allow the derivation of easy-to-compute expressions for the exact average error rate of SSC systems over Rayleigh, Nakagami-n (Rice), and Nakagami-m channels.

  – Results apply to a wide range of modulation schemes.

  – The optimum threshold for the various modulation scheme/fading channel combinations can be found in many instances in closed-form.

  – The presented approach has been extended to study the effect of fading correlation and average fading power imbalance on the performance of SSC systems.

Comparison Between MRC, SC, and SSC

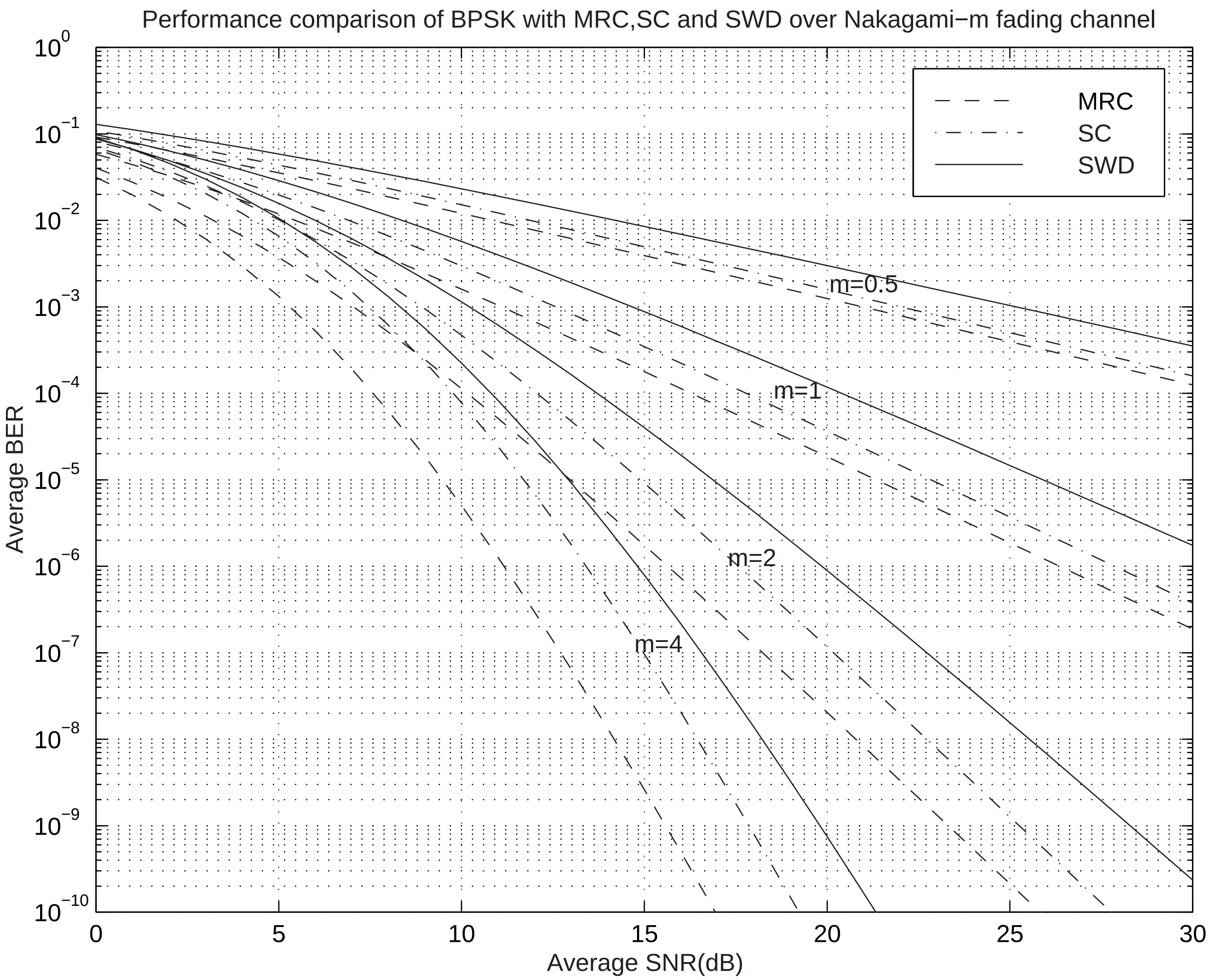


Figure 3: Comparison of the average BER of BPSK with MRC, SC, and SSC (SWD) over Nakagami-m fading channels.

Comparison Between MRC, SC, and SSC

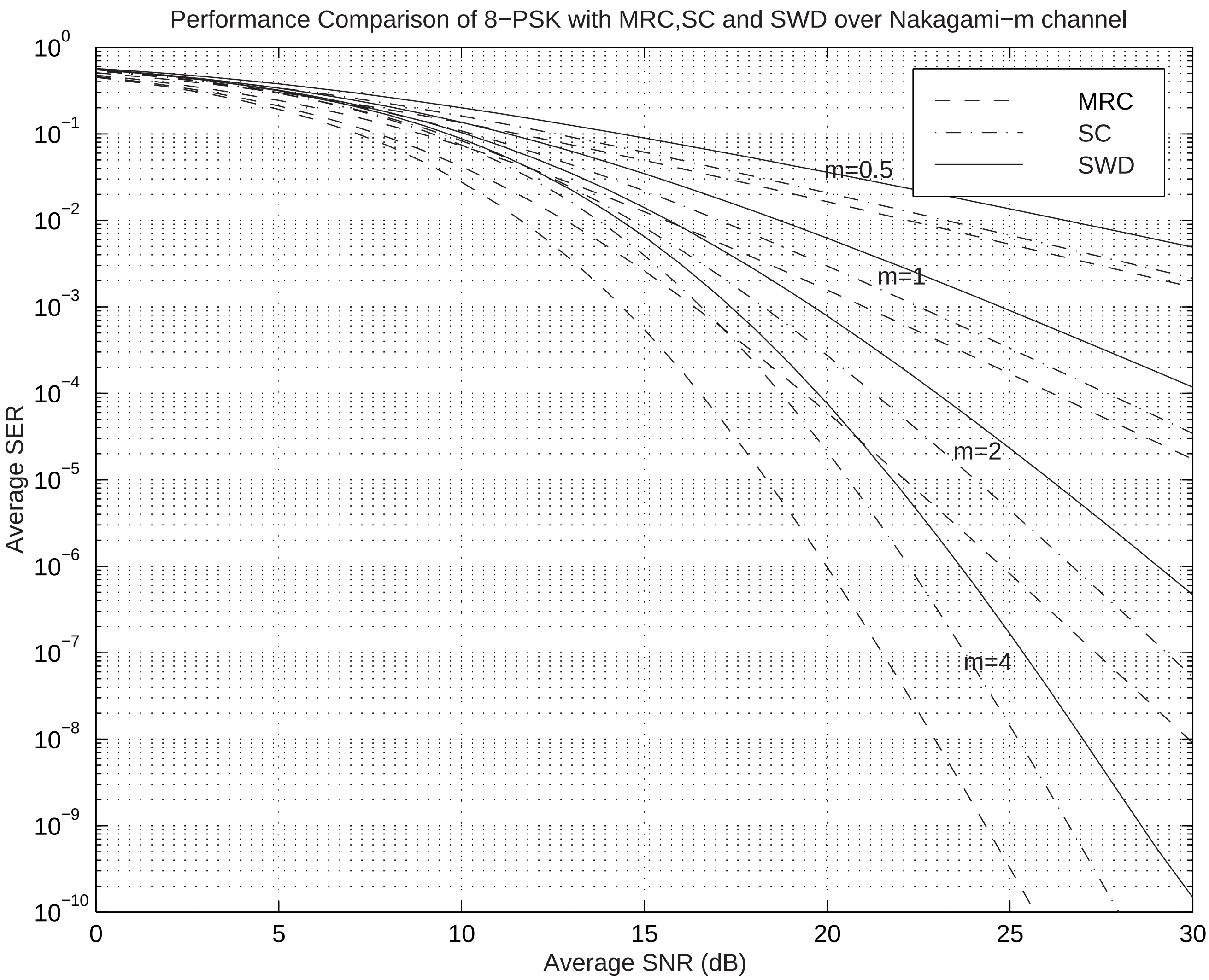


Figure 4: Comparison of the average SER of 8-PSK with MRC, SC, and SSC (SWD) over Nakagami-m fading channels.

Comparison Between MRC, SC, and SSC

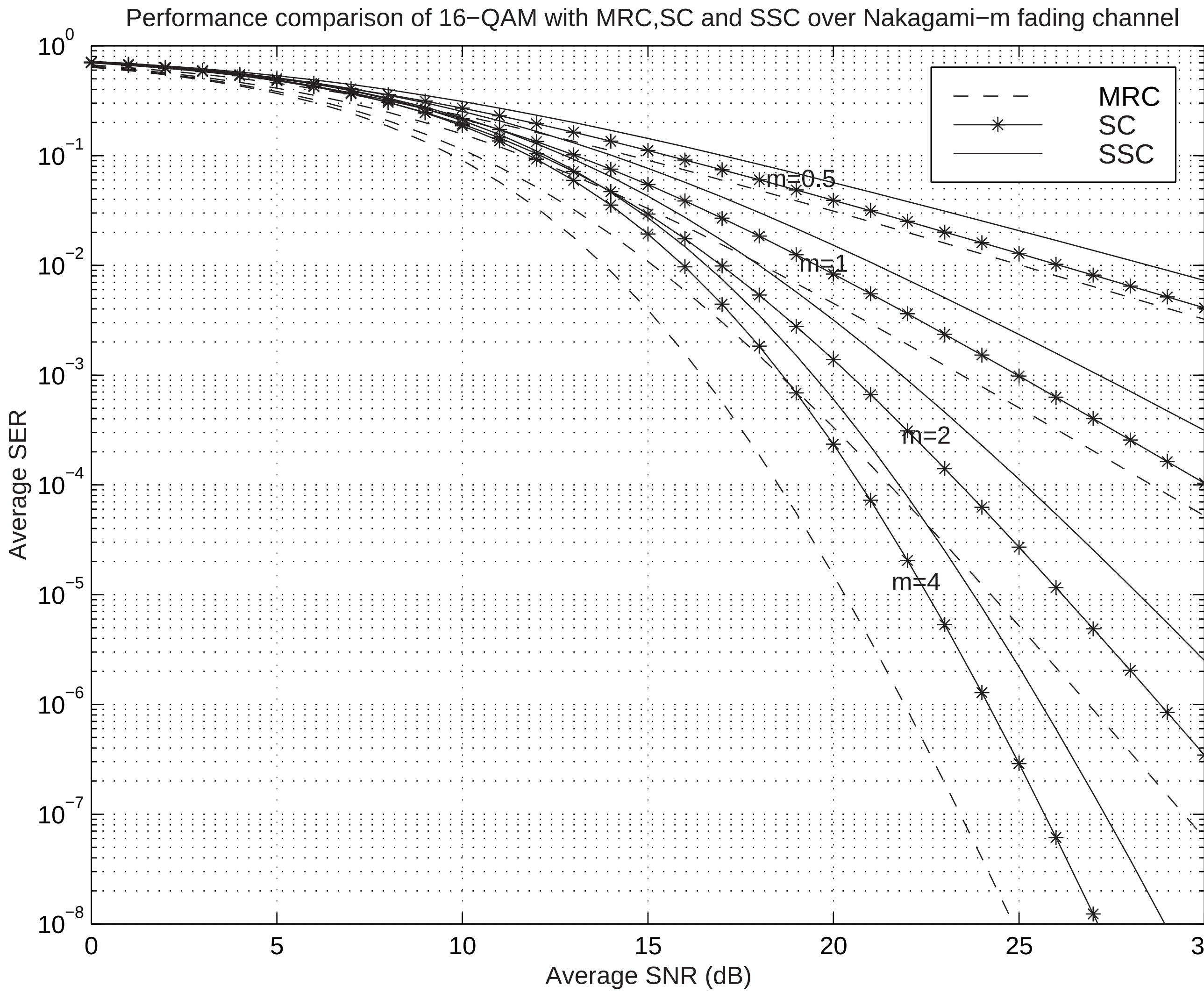


Figure 5: Comparison of the average SER of 16-QAM with MRC, SC, and SSC (SWD) over Nakagami-m fading channels.

Hybrid Diversity Schemes

- Generalized diversity schemes
  - SC/MRC
  - SC/EGC
  - SSC/MRC
  - SSC/EGC
- Two dimensional diversity schemes such as space-multipath diversity (2D-RAKE reception) or frequency-multipath diversity (Multicarrier-RAKE reception).
  - MRC/MRC
  - SC/MRC

Generalized Selection Combining (GSC)

- **Motivation**
  - Complexity of MRC and EGC receivers depends on the number of diversity paths available which is a function of the channel characteristics in the case of multipath diversity.
  - MRC is sensitive to channel estimation errors and these errors tend to be more important when the instantaneous SNR is low.
  - Postdetection noncoherent EGC suffers from combining loss as the number of diversity branches increases.
  - SC uses only one path out of the L available multipaths and hence does not fully exploit the amount of diversity offered by the channel.
  - GSC was introduced as a "bridge" between the two extreme combining techniques offered by SC and MRC (or EGC) by combining the L_c strongest paths among the L available.
  - We denote such a hybrid scheme as SC/MRC-L_c/L or SC/EGC-L_c/L.
- **Goals**
  - Eng, Kong and Milstein studied the average BER of SC/MRC-$2/L$ and SC/MRC-$3/L$ over Rayleigh fading channels by using a PDF-based approach which becomes "extremely unwieldy notationally" for $L_c \geq 4$.
  - We propose to use the MGF-based approach to derive generic expressions valid for any $L_c \leq L$ and for a wide variety of modulation schemes.

GSC Output Statistics

- Let $\gamma_{1:L}, \gamma_{2:L}, \cdots, \gamma_{L:L}$ denote the *order statistics* obtained by arranging the $\{\gamma_l\}_{l=1}^{L}$ in decreasing order of magnitude.
- Assuming that the $\{\gamma_l\}_{l=1}^{L}$ are i.i.d. then the joint PDF of the $\{\gamma_{l:L}\}_{l=1}^{L_c}$ is given by [Papoulis]

$$p_{\gamma_{1:L},\cdots,\gamma_{L_c:L}}(\gamma_{1:L},\cdots,\gamma_{L_c:L})=L_c!\binom{L}{L_c}\left[P_\gamma(\gamma_{L_c:L})\right]^{L-L_c}\prod_{l=1}^{L_c}p_\gamma(\gamma_{l:L}),$$

with

$$\gamma_{1:L} \geq \cdots \geq \gamma_{L_c:L} \geq 0,$$
$$p_\gamma(\gamma) = \frac{1}{\overline{\gamma}}e^{-\frac{\gamma}{\overline{\gamma}}},$$
$$P_\gamma(\gamma) = 1 - e^{-\frac{\gamma}{\overline{\gamma}}}.$$

- It is important to note that although the $\{\gamma_l\}_{l=1}^{L}$ are independent the $\{\gamma_{l:L}\}_{l=1}^{L}$ are not.
- The MGF-based approach relies on finding a simple expression for

$$\mathcal{M}_{\gamma_t}(s)=E_{\gamma_t}[e^{s\gamma_t}] = E_{\gamma_{1:L},\gamma_{2:L},\cdots,\gamma_{L_c:L}}\left[e^{s\sum_{l=1}^{L_c}\gamma_{l:L}}\right]$$
$$=\underbrace{\int_0^\infty\int_{\gamma_{L_c:L}}^\infty\cdots\int_{\gamma_{2:L}}^\infty}_{L_c-\text{fold}} p_{\gamma_{1:L},\cdots,\gamma_{L_c:L}}(\gamma_{1:L},\cdots,\gamma_{L_c:L})e^{s\sum_{l=1}^{L_c}\gamma_{l:L}}d\gamma_{1:L}\cdots d\gamma_{L_c:L}.$$

- **Problem**: Although the integrand is in a desirable separable form in the $\gamma_{l:L_c}$'s, we cannot partition the L_c-fold integral into a product of one-dimensional integrals as was possible for MRC because of the $\gamma_{l:L}$'s in the lower limits of the semi-finite range (improper) integrals.

A Useful Theorem

- **Theorem [Sukhatme 1937]:** Defining the "spacing"

$$\begin{aligned} x_l &\triangleq \gamma_{l:L} - \gamma_{l+1:L} \quad (l = 1, 2, \cdots, L-1) \\ x_L &\triangleq \gamma_{L:L} \end{aligned}$$

then the $\{x_l\}_{l=1}^{L}$ are

  - Independently distributed
  - Distributed according to an exponential distribution

$$p_{x_l}(x_l) = \frac{l}{\overline{\gamma}} e^{-\frac{l x_l}{\overline{\gamma}}}, \quad x_l \geq 0, (l = 1, 2, \cdots, L)$$

- **Sketch of the Proof:**
  - Since the Jacobian of the transformation is equal to 1 we have

$$\begin{aligned} p_{x_1, \cdots, x_L}(x_1, \cdots, x_L) &= p_{\gamma_{1:L}, \cdots, \gamma_{L:L}}(\gamma_{1:L}, \cdots, \gamma_{L:L}) \\ &= \frac{L!}{\overline{\gamma}^L} \exp\left(-\frac{\sum_{l=1}^{L} \gamma_{l:L}}{\overline{\gamma}} \right). \end{aligned}$$

  - The $\gamma_{l:L}$'s can be expressed in terms of the x_l's as

$$\gamma_{l:L} = \sum_{k=l}^{L} x_k.$$

  - Hence

$$\begin{aligned} p_{x_1, \cdots, x_L}(x_1, \cdots, x_L) &= \frac{L!}{\overline{\gamma}^L} \exp\left(-\frac{x_1 + 2x_2 + \cdots + L x_L}{\overline{\gamma}} \right) \\ &= \prod_{l=1}^{L} \frac{l}{\overline{\gamma}} \exp\left(-\frac{l x_l}{\overline{\gamma}} \right). \quad \text{QED} \end{aligned}$$

MGF of GSC Combined SNR

- We use the previous theorem to derive a simple expression for the MGF of the combined SNR γ_t given by

$$\begin{aligned}\gamma_t &= \sum_{l=1}^{L_c} \gamma_{l:L} = \sum_{l=1}^{L_c} \sum_{k=l}^{L} x_k \\ &= x_1 + 2x_2 + \cdots + L_c x_{L_c} + L_c x_{L_c+1} + \cdots + L_c x_L.\end{aligned}$$

1. Rewriting the MGF of γ_t in terms of the x_l's as

$$\mathcal{M}_{\gamma_t}(s) = \underbrace{\int_0^\infty \cdots \int_0^\infty}_{L-\text{fold}} p_{x_1,\cdots,x_L}(x_1,\cdots,x_L) e^{s(x_1+2x_2+\cdots+L_c x_{L_c}+L_c x_{L_c+1}+\cdots+L_c x_L)} dx_1 \cdots dx_L.$$

2. Since the x_l's are independent $p_{x_1,\cdots,x_L}(x_1,\cdots,x_L) = \prod_{l=1}^{L} p_{x_l}(x_l)$ and we can hence put the integrand in a product form

$$\mathcal{M}_{\gamma_t}(s) = \underbrace{\int_0^\infty \cdots \int_0^\infty}_{L-\text{fold}} \left[\prod_{l=1}^{L} p_{x_l}(x_l)\right] e^{sx_1} e^{2sx_2} \cdots e^{L_c s x_{L_c}} e^{L_c s x_{L_c+1}} \cdots e^{L_c s x_L} dx_1 \cdots dx_L.$$

3. Grouping like terms and partitioning the L-fold integral into a product of L one-dimensional integrals

$$\begin{aligned}\mathcal{M}_{\gamma_t}(s) = &\left[\int_0^\infty e^{sx_1} p_{x_1}(x_1) dx_1\right]\left[\int_0^\infty e^{2sx_2} p_{x_2}(x_2) dx_2\right] \cdots \left[\int_0^\infty e^{L_c s x_{L_c}} p_{x_{L_c}}(x_{L_c}) dx\right. \\ &\times \left[\int_0^\infty e^{L_c s x_{L_c+1}} p_{x_{L_c+1}}(x_{L_c+1}) dx_{L_c+1}\right] \cdots \left[\int_0^\infty e^{L_c s x_L} p_{x_L}(x_L) dx_L\right].\end{aligned}$$

4. Using the fact that the x_l's are exponentially distributed we get the final desired closed-form result as

$$\mathcal{M}_{\gamma_t}(s) = (1 - s\overline{\gamma})^{-L_c} \prod_{l=L_c+1}^{L} \left(1 - \frac{s\overline{\gamma} L_c}{l}\right)^{-1}.$$

Average Combined SNR of GSC

- Cumulant generating function at the GSC output is

$$\Psi_{\gamma_{\rm gsc}}(s) = \ln(\mathcal{M}_{\gamma_{\rm gsc}}(s)) = -L_c \ \ln(1 - s\overline{\gamma}) - \sum_{l=L_c+1}^{L} \ln\left(1 - \frac{s\overline{\gamma}L_c}{l}\right).$$

- The first cumulant of $\gamma_{\rm gsc}$ is equal to its statistical average:

$$\overline{\gamma}_{\rm gsc} = \left.\frac{d\Psi_{\gamma_{\rm gsc}}(s)}{ds}\right|_{s=0},$$

giving [Kong and Milstein 98]

$$\overline{\gamma}_{\rm gsc} = \left(1 + \sum_{l=L_c+1}^{L} \frac{1}{l}\right) L_c\overline{\gamma}.$$

- Generalizes the average SNR results for conventional SC and MRC:
  - For $L = L_c$, $\overline{\gamma}_{\rm mrc} = L\overline{\gamma}$.
  - For $L_c = 1$, $\overline{\gamma}_{\rm sc} = \sum_{l=1}^{L} \frac{1}{l}\ \overline{\gamma}$

Average Combined SNR of GSC

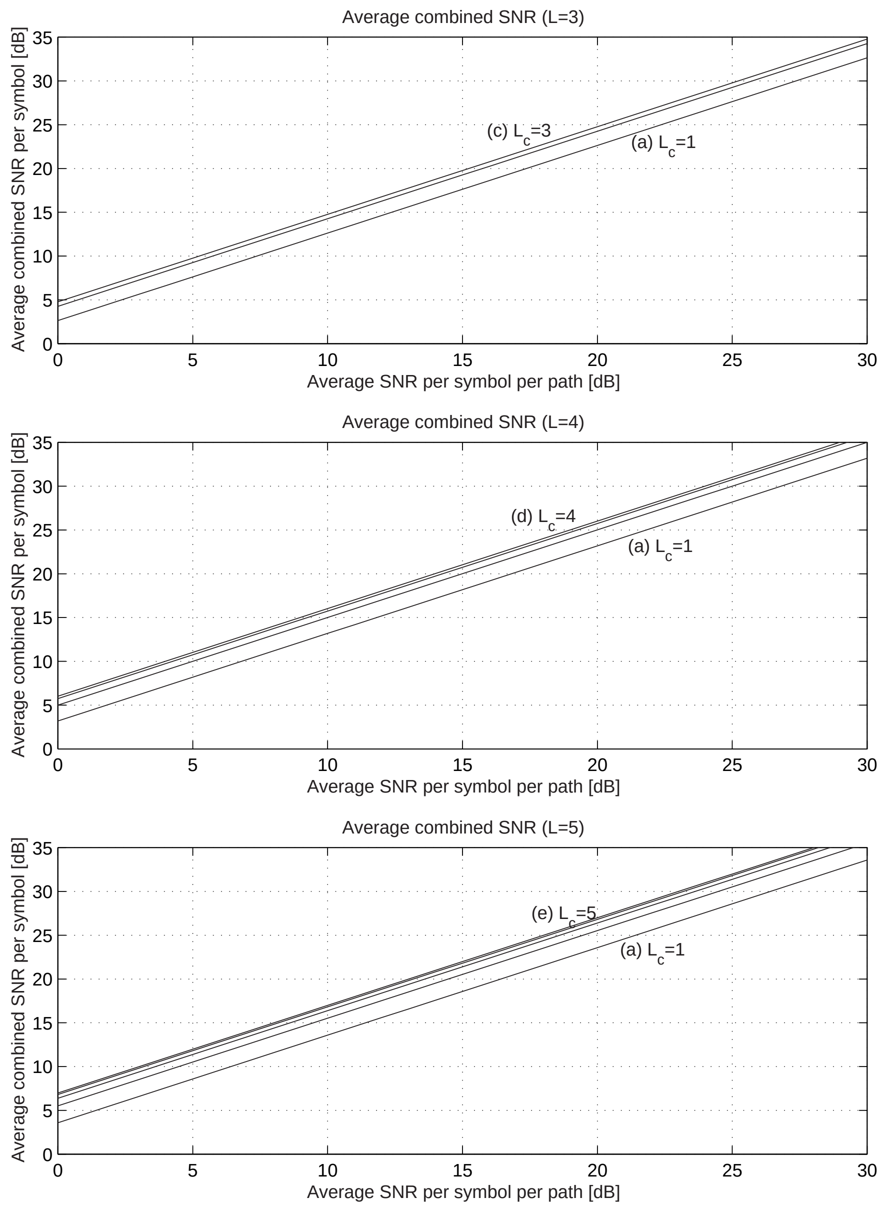


Figure 6: Average combined signal-to-noise ratio (SNR) $\overline{\gamma}_{\mathrm{gsc}}$ versus the average SNR per path $\overline{\gamma}$ for A- $L = 3$ ((a) $L_c = 1$ (SC), (b) $L_c = 2$, and (c) $L_c = 3$ (MRC)), B- $L = 4$ ((a) $L_c = 1$ (SC), (b) $L_c = 2$, (c) $L_c = 3$, and (d) $L_c = 4$ (MRC)), and C- $L = 5$ ((a) $L_c = 1$ (SC), (b) $L_c = 2$, (c) $L_c = 3$, (d) $L_c = 4$, and (e) $L_c = 5$ MRC)).

Average Combined SNR of GSC

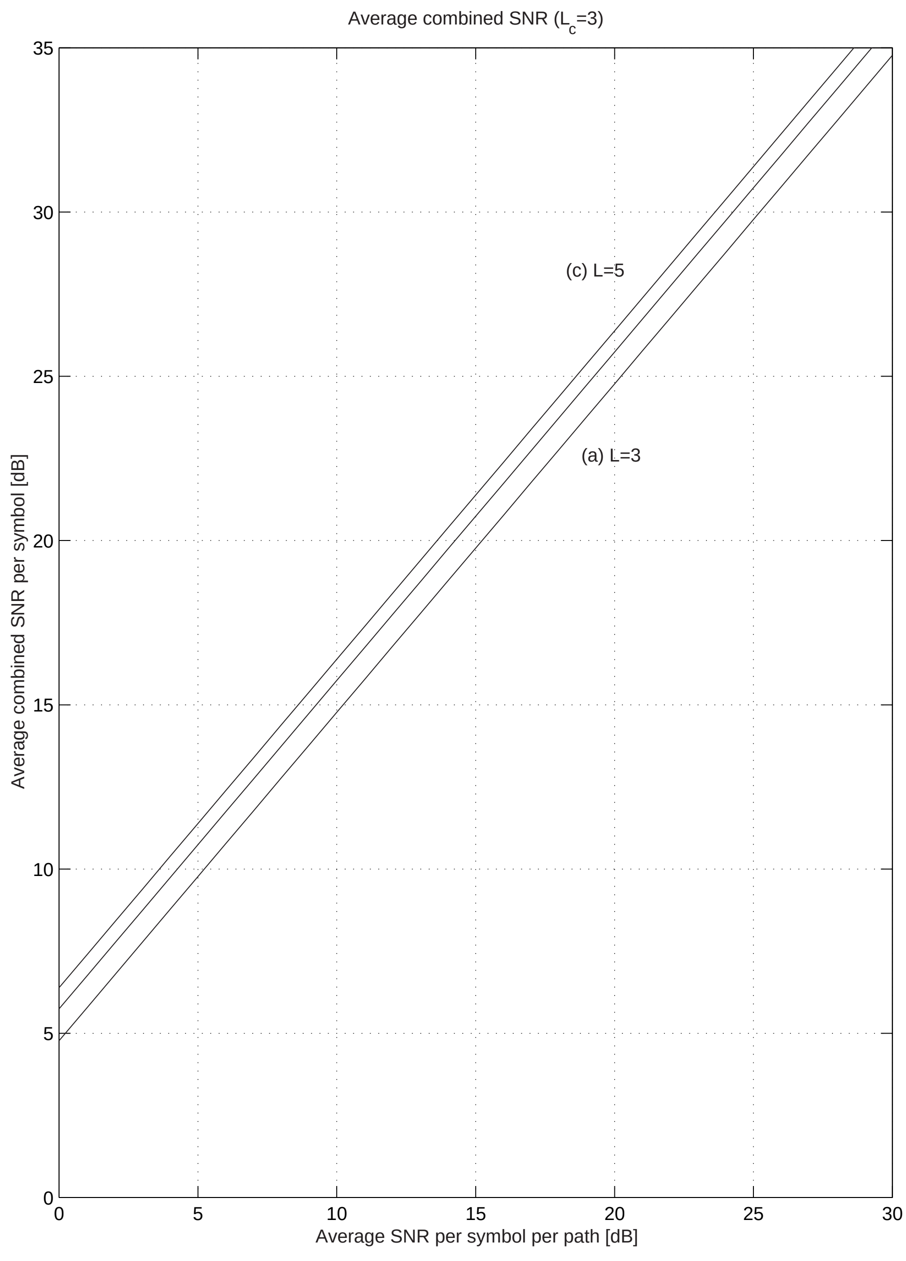


Figure 7: Average combined signal-to-noise ratio (SNR) $\overline{\gamma}_{\mathrm{gsc}}$ versus the average SNR per path $\overline{\gamma}$ for $L_c = 3$ ((a) $L = 3$, (b) $L = 4$, and (c) $L = 5$.

Performance of 16-QAM with GSC

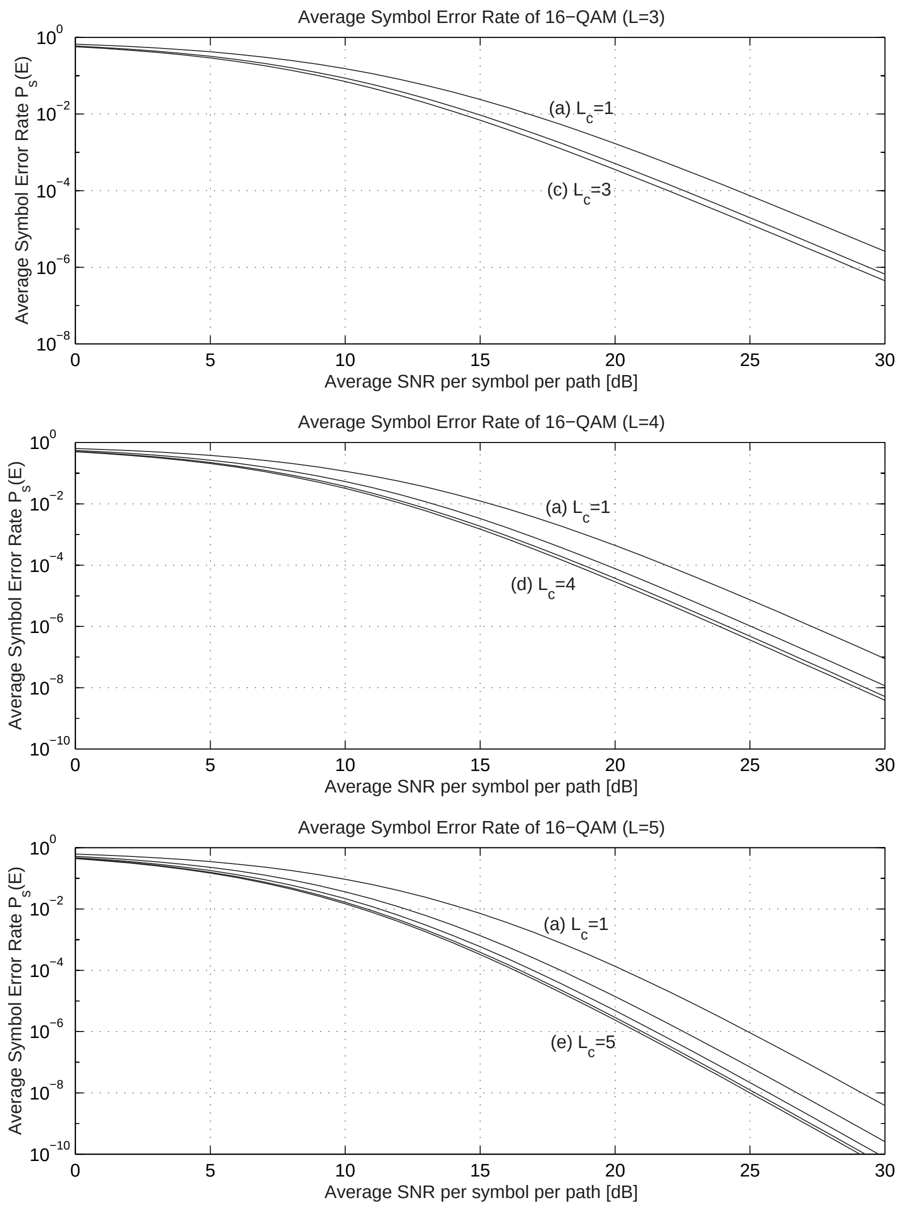


Figure 8: Average symbol error rate (SER) $P_s(E)$ of 16-QAM versus the average SNR per symbol per path $\overline{\gamma}$ for A- $L = 3$ ((a) $L_c = 1$ (SC), (b) $L_c = 2$, and (c) $L_c = 3$ (MRC)), B- $L = 4$ ((a) $L_c = 1$ (SC), (b) $L_c = 2$, (c) $L_c = 3$, and (d) $L_c = 4$ (MRC)), and C- $L = 5$ ((a) $L_c = 1$ (SC), (b) $L_c = 2$, (c) $L_c = 3$, (d) $L_c = 4$, and (e) $L_c = 5$ MRC)).

Performance of 16-QAM with GSC

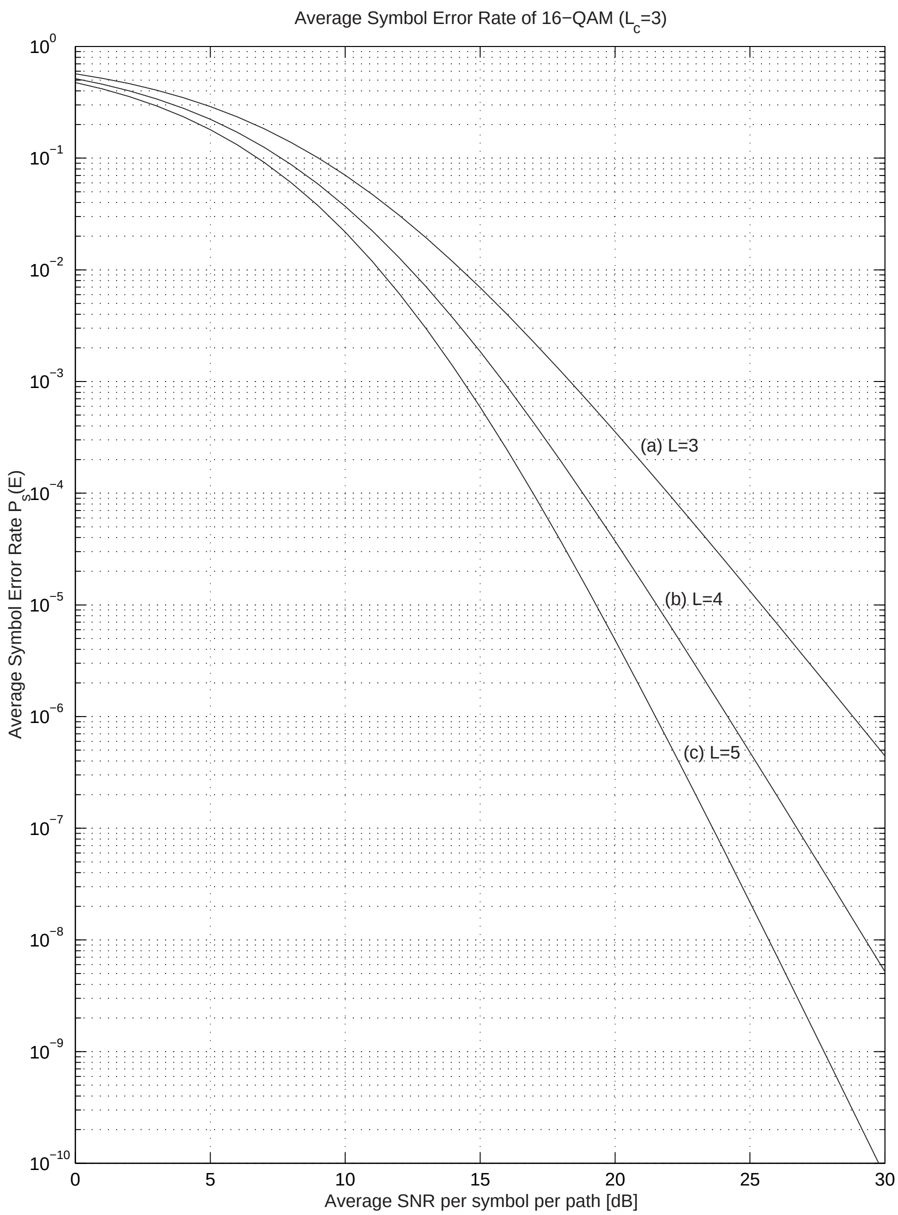


Figure 9: Average symbol error rate (SER) $P_s(E)$ of 16-QAM versus the average SNR per symbol per path $\overline{\gamma}$ for $L_c = 3$ ((a) $L = 3$, (b) $L = 4$, and (c) $L = 5$).

Impact of Correlation on the Performance of MRC Diversity Systems

- **Motivation**
  - In some real life scenarios the independence assumption is not valid (e.g. insufficient antenna spacing in small-size mobile units equipped with space antenna diversity).
  - In correlated fading conditions the maximum theoretical diversity gain cannot be achieved.
  - Effect of correlation between the combined signals has to be taken into account for the accurate performance analysis of diversity systems.
- **Goal**
  - Obtain generic easy-to-compute formulas for the exact average error probability in correlated fading environment:
    * Accounting for the average SNR imbalance and severity of fading (Nakagami-m).
    * A variety of correlation models.
    * Wide range of modulation schemes.
- **Tools**
  - The unified moment generating function (MGF) based approach.
  - Mathematical studies on the multivariate gamma distribution (Krishnamoorthy and Parthasarathy 51, Gurland 55, and Kotz and Adams 64).

Summary of the MGF-based Approach

- **MGF-Based Approach**
  - Uses alternate representations of classic functions such as Gaussian Q-function and Marcum Q-function.
  - Finds alternate representation of the conditional error rate

$$P_s(E/\gamma_t) = \sum \int_{\theta_1}^{\theta_2} h(\phi) e^{-g(\phi)\gamma_t} \, d\phi$$

  - Switching order of integration is possible

$$\begin{aligned} P_s(E) &= \sum \int_{\theta_1}^{\theta_2} h(\phi) \underbrace{\int_0^{\infty} p_{\gamma_t}(\gamma_t) \, e^{-g(\phi)\gamma_t} \, d\gamma_t}_{\mathcal{M}(-g(\phi))} \, d\phi \\ &= \sum \int_{\theta_1}^{\theta_2} h(\phi) \, \mathcal{M}(-g(\phi)) \, d\phi, \end{aligned}$$

    where

$$\mathcal{M}(s) \overset{\triangle}{=} E_{\gamma_t}\left[e^{s\gamma_t}\right] = \int_0^{\infty} p_{\gamma_t}(\gamma_t) \, e^{s\gamma_t} \, d\gamma_t.$$

- **Example**
  - Average symbol error rate (SER) of M-PSK signals

$$P_s(E) = \frac{1}{\pi} \int_0^{\frac{(M-1)\pi}{M}} \mathcal{M}\left(-\frac{\sin^2\left(\frac{\pi}{M}\right)}{\sin^2 \phi} \right) \, d\phi$$

Model A: Dual Diversity

- Two correlated branches with nonidentical fading (e.g. polarization diversity).
- PDF of the combined SNR

$$p_a(\gamma_t) = \frac{\sqrt{\pi}}{\Gamma(m)} \left[\frac{m^2}{\overline{\gamma}_1\overline{\gamma}_2(1-\rho)}\right]^m \left(\frac{\gamma_t}{2\beta'}\right)^{m-\frac{1}{2}} I_{m-\frac{1}{2}}(\beta'\gamma_t)\, e^{-\alpha'\gamma_t}; \;\; \gamma_t \geq 0,$$

where

$$\rho = \frac{\text{cov}(r_1^2, r_2^2)}{\sqrt{\text{var}(r_1^2)\text{var}(r_2^2)}}, \;\; 0 \leq \rho < 1.$$

is the envelope correlation coefficient between the two signals, and

$$\alpha' \triangleq \frac{\alpha}{E_s/N_0} = \frac{m(\overline{\gamma}_1 + \overline{\gamma}_2)}{2\overline{\gamma}_1\overline{\gamma}_2(1-\rho)},$$

$$\beta' \triangleq \frac{\beta}{E_s/N_0} = \frac{m\left((\overline{\gamma}_1 + \overline{\gamma}_2)^2 - 4\overline{\gamma}_1\overline{\gamma}_2(1-\rho)\right)^{1/2}}{2\overline{\gamma}_1\overline{\gamma}_2(1-\rho)}.$$

- MGF of the combined SNR per symbol

$$\mathcal{M}_a(s) = \left(1 - \frac{(\overline{\gamma}_1 + \overline{\gamma}_2)}{m}\, s + \frac{(1-\rho)\overline{\gamma}_1\overline{\gamma}_2}{m^2}\, s^2\right)^{-m}; \;\; s \geq 0.$$

- With this model for BPSK the MGF-based approach gives an alternate form to the previous equivalent result [Aalo 95] which required the evaluation of the Appell's hypergeometric function, $F_2(\cdot;\cdot,\cdot;\cdot,\cdot;\cdot,\cdot)$.

Model B: Multiple Diversity with Constant Correlation

- D identically distributed Nakagami-m channels with constant correlation
  - Same average SNR/symbol/channel $\overline{\gamma}_d = \overline{\gamma}$ and the same fading parameter m.
  - Envelope correlation coefficient ρ is the same between all the channel pairs.
- Corresponds for example to the scenario of multichannel reception from closely placed diversity antennas.
- PDF of the combined SNR

$$p_b(\gamma_t)=\frac{\left(\frac{m\gamma_t}{\overline{\gamma}}\right)^{Dm-1} \exp\left(-\frac{m\gamma_t}{(1-\sqrt{\rho})\overline{\gamma}}\right)\ {}_1F_1\left(m, Dm; \frac{Dm\sqrt{\rho}\gamma_t}{(1-\sqrt{\rho})(1-\sqrt{\rho}+D\sqrt{\rho})\overline{\gamma}}\right)}{\left(\frac{\overline{\gamma}}{m}\right)\ (1-\sqrt{\rho})^{m(D-1)}\ (1-\sqrt{\rho}+D\sqrt{\rho})^{m}\ \Gamma(Dm)};\ \ \gamma_t \geq 0.$$

  where ${}_1F_1(\cdot,\cdot;\cdot)$ is the confluent hypergeometric function.

- MGF of the combined SNR per symbol

$$\mathcal{M}_b(s)=\left(1-\frac{\overline{\gamma}(1-\sqrt{\rho}+D\sqrt{\rho})}{m}\ s\right)^{-m}\ \left(1-\frac{\overline{\gamma}(1-\sqrt{\rho})}{m}\ s\right)^{-m(D-1)};\ \ s \geq 0.$$

Model C: Multiple Diversity with Arbitrary Correlation

- D identically distributed Nakagami-m channels with arbitrary correlation.
  - Same average SNR/symbol/channel $\overline{\gamma}_d = \overline{\gamma}$ and the same fading parameter m.
  - Envelope correlation coefficient $\rho_{dd'}$ may be different between the channel pairs.
- Useful for example to the scenario of multichannel reception from diversity antennas in which the correlation between the pairs of combined signals decays as the spacing between the antennas increases.
- PDF of the combined SNR not available in a simple form.
- MGF of the combined SNR per symbol can be deduced from the work of [Krishnamoorthy and Parthasarathy 51]

$$
\begin{aligned}
\mathcal{M}_c(s) &= E_{\gamma_1,\gamma_2,\cdots,\gamma_D}\left[\exp\left(s\sum_{d=1}^{D}\gamma_d\right)\right] \\
&= \left(-\frac{s\overline{\gamma}}{m}\right)^{-mD}\left|\begin{bmatrix} 1-\frac{m}{s\overline{\gamma}} & \sqrt{\rho_{12}} & \cdots & \sqrt{\rho_{1D}} \\ \sqrt{\rho_{12}} & 1-\frac{m}{s\overline{\gamma}} & \cdots & \sqrt{\rho_{2D}} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \sqrt{\rho_{1D}} & \sqrt{\rho_{2D}} & \cdots & 1-\frac{m}{s\overline{\gamma}} \end{bmatrix}\right|_{D\times D}^{-m},
\end{aligned}
$$

where $|[M]|_{D\times D}$ denotes the determinant of the $D \times D$ matrix M.

Special Cases of Model C

- Dual Correlation Model (Model A)
  - A dual correlation model ($D = 2$) has a correlation matrix with the following structure
$$M = \begin{bmatrix} 1 - \frac{m}{s\overline{\gamma}} & \sqrt{\rho} \\ \sqrt{\rho} & 1 - \frac{m}{s\overline{\gamma}} \end{bmatrix}.$$
  - Application: Small size terminals equipped with space diversity where antenna spacing is insufficient to provide independent fading among signal paths.
  - The determinant of M can be easily found to be given by
$$\det M = \left(1 - \frac{m}{s\overline{\gamma}}\right)^2 - \rho.$$
  - Substituting the determinant of M in the MGF we get
$$\mathcal{M}_c(s) = \mathcal{M}_a(s) = \left(1 - \frac{2\overline{\gamma}}{m}s + \frac{(1-\rho)\overline{\gamma}^2}{m^2}s^2\right)^{-m}.$$

- Intraclass Correlation Model (Model B)
  - A correlation matrix M is called a Dth order intraclass correlation matrix iff it has the following structure

$$M = \begin{bmatrix} a & b & \cdot & \cdot & \cdot & b \\ b & a & b & \cdot & \cdot & b \\ b & b & a & b & \cdot & b \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & b & a \end{bmatrix}_{D \times D}$$

    with $b \geq -\frac{a}{D-1}$.
  - Application: Very closely spaced antennas or 3 antennas placed on an equilateral triangle.
  - Theorem: If M is a Dth order intraclass correlation matrix then

$$\det M = (a-b)^{D-1}\,(a + b(D-1))$$

  - For $a = 1 - \frac{m}{s\overline{\gamma}}$ and $b = \sqrt{\rho}$, applying the previous theorem we get

$$\mathcal{M}_c(s) = \mathcal{M}_b(s) = \left(1 - \frac{\overline{\gamma}(1-\sqrt{\rho}+D\sqrt{\rho})}{m}\ s\right)^{-m} \left(1 - \frac{\overline{\gamma}(1-\sqrt{\rho})}{m}\ s\right)^{-m(D-1)}.$$

- Exponential Correlation Model
  - An exponential correlation model is characterized by $\rho_{dd'} = \rho^{|d-d'|}$.
  - Application: correspond for example to the scenario of multichannel reception from equispaced diversity antennas in which the correlation between the pairs of combined signals decays as the spacing between the antennas increases.
  - Using the algebraic technique presented in [Pierce 60] it can be easily shown that the MGF is in this case given by

$$\mathcal{M}_c(s) = \left(-\frac{s\overline{\gamma}}{m}\right)^{-mD} \prod_{d=1}^{D} \left(\frac{1-\rho}{1+\rho+2\sqrt{\rho}\cos\theta_d}\right)^{-m},$$

where θ_d ($d = 1, 2, 3, \cdots, D$) are the D solutions of the transcendental equation given by

$$\tan(D\theta_d) = \frac{-\sin\theta_d}{\left(\frac{1+\rho}{1-\rho}\right)\cos\theta_d + \frac{2\sqrt{\rho}}{1-\rho}}.$$

- Tridiagonal Correlation Model
  - A correlation matrix M is called a Dth order tridiagonal correlation matrix iff it has the following structure

$$M = \begin{bmatrix} a & b & 0 & \cdot & \cdot & 0 \\ b & a & b & 0 & \cdot & 0 \\ 0 & b & a & b & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & b & a \end{bmatrix}_{D \times D}$$

  - Application: A "nearly" perfect antenna array in which the signal received at any antenna is weakly correlated with that received at any adjacent antenna, but beyond adjacent antenna the correlation is zero.
  - Theorem: If M is a Dth order tridiagonal correlation matrix then

$$\det M = \prod_{d=1}^{D} \left(a + 2b \cos\left(\frac{d\pi}{D+1} \right) \right)$$

  - For $a = 1 - \frac{m}{s\overline{\gamma}}$ and $b = \sqrt{\rho}$, applying the previous Theorem we get

$$\mathcal{M}_c(s) = \prod_{d=1}^{D} \left(1 - \frac{s\overline{\gamma}}{m} \left(1 + 2\sqrt{\rho} \cos\left(\frac{d\pi}{D+1} \right) \right) \right)^{-m}$$

    with

$$\rho \leq \frac{1}{4\cos^2\left(\frac{\pi}{D+1}\right)},$$

    to insure that the matrix M is nonsingular and nonnegative.

Effect of Correlation on 8-PSK

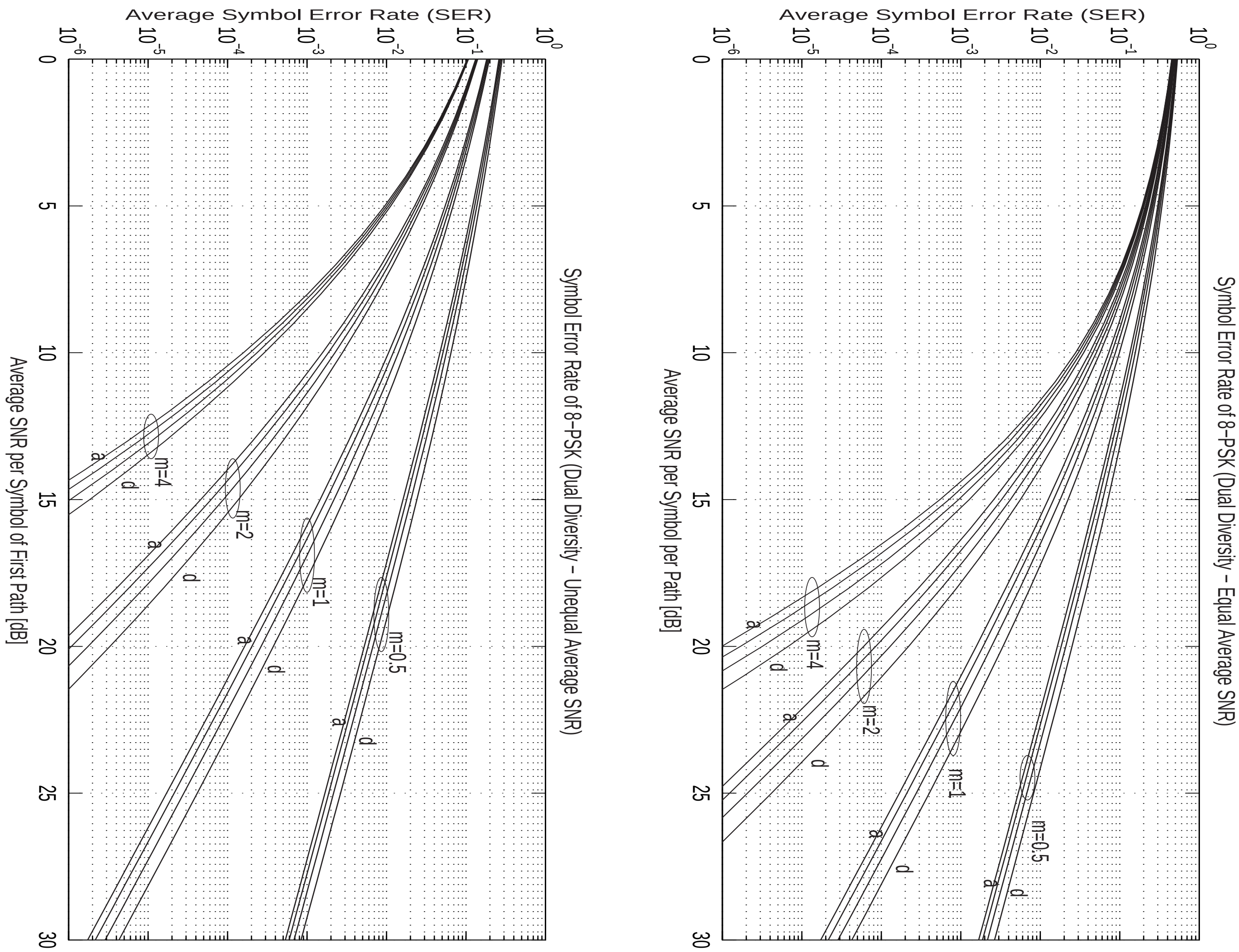


Figure 10: Average SER of 8-PSK with dual MRC diversity for various values of the correlation coefficient ((a) $\rho = 0$, (b) $\rho = 0.2$, (c) $\rho = 0.4$, and (d) $\rho = 0.6$) and for A- Equal average branch SNRs ($\overline{\gamma}_1 = \overline{\gamma}_2$) and B- Unequal average branch SNRs ($\overline{\gamma}_1 = 10\ \overline{\gamma}_2$).

Effect of Correlation on 8-PSK

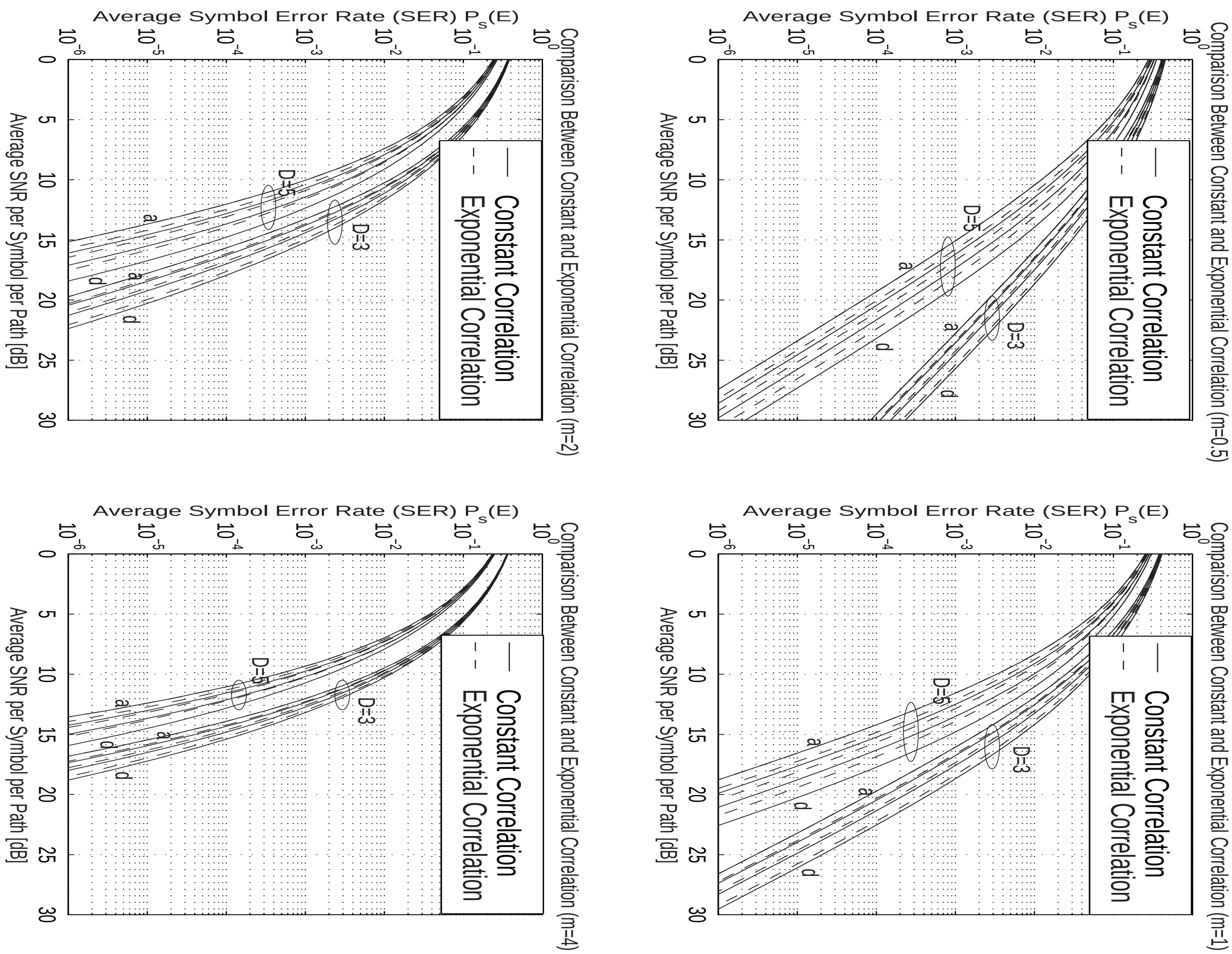


Figure 11: Comparison of the average SER of 8-PSK with MRC diversity for constant and exponential fading correlation profiles, various values of the correlation coefficient ((a) $\rho = 0$, (b) $\rho = 0.2$, (c) $\rho = 0.4$, and (d) $\rho = 0.6$), and $\overline{\gamma}_d = \overline{\gamma}$ for $d = 1, 2, \cdots, D$.

2D-MRC/MRC Diversity over Correlated Fading

- We consider a two-dimensional diversity system consisting for example of D antennas each one followed by an L_c finger RAKE receiver.
- For practical channel conditions of interest we have
  - For a fixed antenna index d assume that the $\{\gamma_{l,d}\}_{l=1}^{L_c}$'s are independent but nonidentically distributed.
  - For a fixed multipath index l assume that the $\{\gamma_{l,d}\}_{d=1}^{D}$'s are correlated according to model A, B, or C (as described earlier).
- When MRC combining is done for both space and multipath diversity we have a conditional combined SNR/bit given by

$$\begin{aligned} \gamma_t &= \sum_{d=1}^{D} \sum_{l=1}^{L_c} \gamma_{l,d} \\ &= \sum_{d=1}^{D} \gamma_d \text{ (where } \gamma_d = \sum_{l=1}^{L_c} \gamma_{l,d}) \\ &= \sum_{l=1}^{L_c} \gamma_l \text{ (where } \gamma_l = \sum_{d=1}^{D} \gamma_{l,d}). \end{aligned}$$

- Finding the average error rate performance of such systems with the classical PDF-based approach is difficult since the PDF of γ_t cannot be found in a simple form.
- We propose to use the MGF-based approach to obtain generic results for a wide variety of modulation schemes.

MGF-Based Approach for 2D-MRC/MRC Diversity over Correlated Fading

- Using the MGF-based approach for the average BER of BPSK we have after switching order of integration

$$P_b(E) = \frac{1}{\pi}\int_0^{\pi/2} E_{\gamma_1,\gamma_2,\cdots,\gamma_{L_c}}\left[\exp\left(\frac{-\sum_{l=1}^{L_c}\gamma_l}{\sin^2\phi}\right)\right]\, d\phi.$$

- Since the $\{\gamma_l\}_{l=1}^{L_c}$ are assumed to be independent then

$$\begin{aligned} P_b(E) &= \frac{1}{\pi}\int_0^{\pi/2}\prod_{l=1}^{L_c} E_{\gamma_l}\left[\exp\left(-\frac{\gamma_l}{\sin^2\phi}\right)\right]\, d\phi \\ &= \frac{1}{\pi}\int_0^{\pi/2}\prod_{l=1}^{L_c}\mathcal{M}\gamma_l\left(-\frac{1}{\sin^2\phi}\right)\, d\phi. \end{aligned}$$

- **Example:**
  - Assume constant correlation ρ_l along the path of index l ($l = 1, 2, \cdots, L_c$) (correlation model B).
  - Assume the same exponential power delay profile in the D RAKE receivers:

    $$\overline{\gamma}_{l,d} = \overline{\gamma}_{1,1}\, e^{-(l-1)\delta}\ \ (l = 1, 2, \cdots, L_c),$$

    where δ is average fading power decay factor.
  - Average BER for BPSK with the MGF-based approach:

$$P_b(E) = \frac{1}{\pi}\int_0^{\pi/2}\prod_{l=1}^{L_c}\left(1 + \frac{\overline{\gamma}_{l,d}(1-\sqrt{\rho_l}+D\sqrt{\rho_l})}{m_l\ \sin^2\phi}\right)^{-m_l}\left(1 + \frac{\overline{\gamma}_{l,d}(1-\sqrt{\rho_l})}{m_l\ \sin^2\phi}\right)^{-m_l(D-1)} d\phi.$$

BER Performance of 2D RAKE Receivers

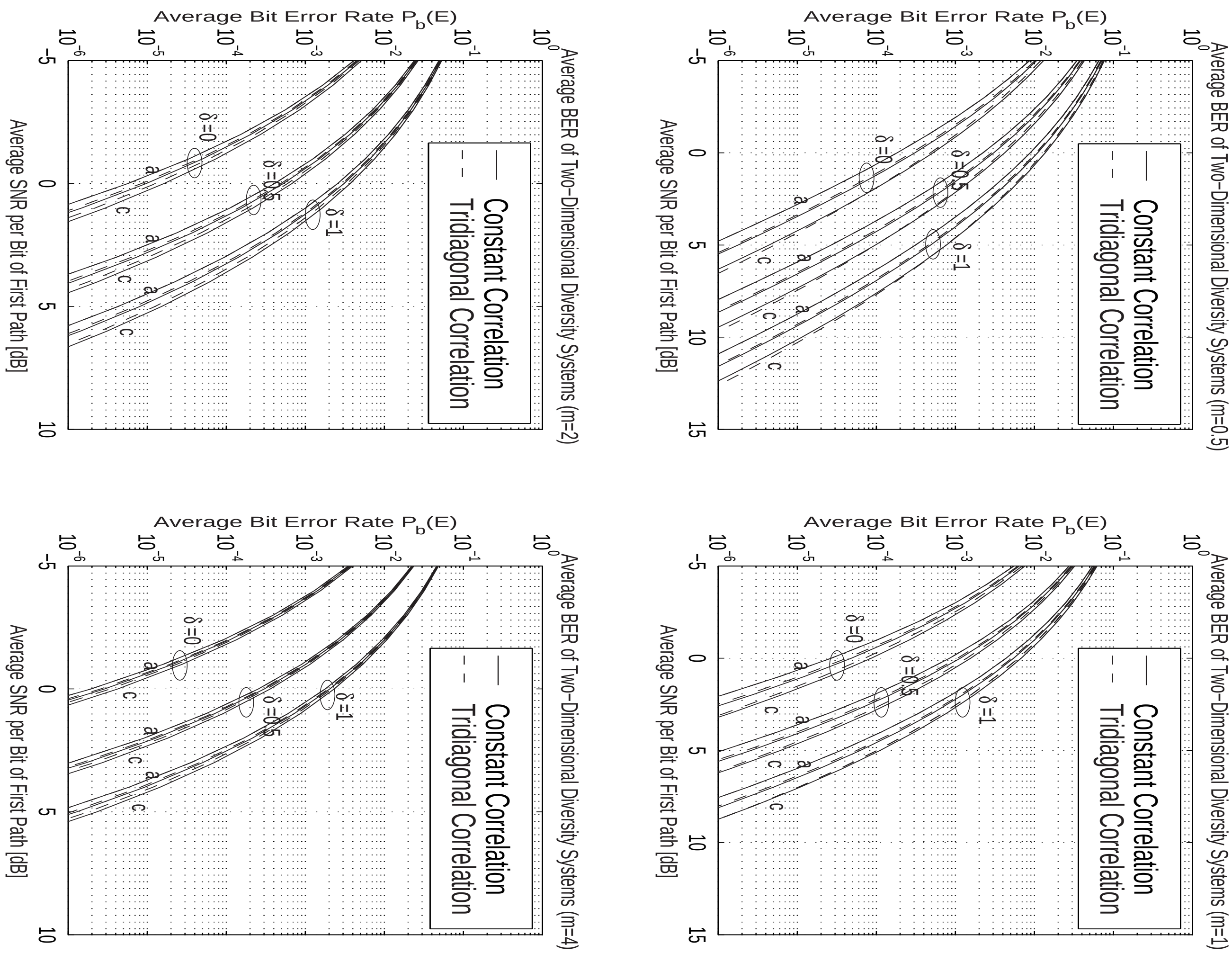


Figure 12: Average BER of BPSK with 2D MRC RAKE reception ($L_c = 4$ and $D = 3$) over an exponentially decaying power delay profile and constant or tridiagonal spatial correlation between the antennas for various values of the correlation coefficient ((a) $\rho = 0$, (b) $\rho = 0.2$, (c) $\rho = 0.4$).

Optimal Transmitter Diversity

- Approximated BER of M-QAM and M-PSK

$$P_b(E|\gamma) = a \cdot \exp(-b\gamma).$$

For example, $a = 0.0852$ and b=0.4030 for 16-QAM.

- Average BER with MRC combining
  - Average BER

$$P_b(E) = a \prod_{l=1}^{L} \left(1 + \frac{b\overline{\gamma_l}}{m_l}\right)^{-m_l},$$

where $\overline{\gamma}_l = \frac{\Omega_l E_s^{(l)}}{N_l} = \frac{\Omega_l P_l T_s}{N_l} = P_l G_l$.

  - **Goal:** Find the set $\{P_l\}_{l=1}^{L}$ which minimizes the average BER subject to the total power constraint $P_t = \sum_{l=1}^{L} P_l$.
  - There exists a unique optimal power allocation solution
    - The constraint forms a convex set.
    - $\frac{a}{P_b(E)}$ is concave.

Optimal Solution

- Optimum power for minimum average BER

$$P_l = m_l \;\text{Max}\left[\frac{P_t}{\sum_{k=1}^{L} m_k} + \frac{\sum_{k=1}^{L}\frac{m_k}{G_k}}{b\sum_{k=1}^{L} m_k} - \frac{1}{bG_l}, 0\right].$$

- For all equal Nakagami parameter m

$$P_l = \text{Max}\left[\frac{P_t}{L} + \frac{m}{Lb}\sum_{k=1}^{L}\frac{1}{G_k} - \frac{m}{bG_l}, 0\right].$$

- For the Rayleigh fading channel (i.e., $m = 1$)

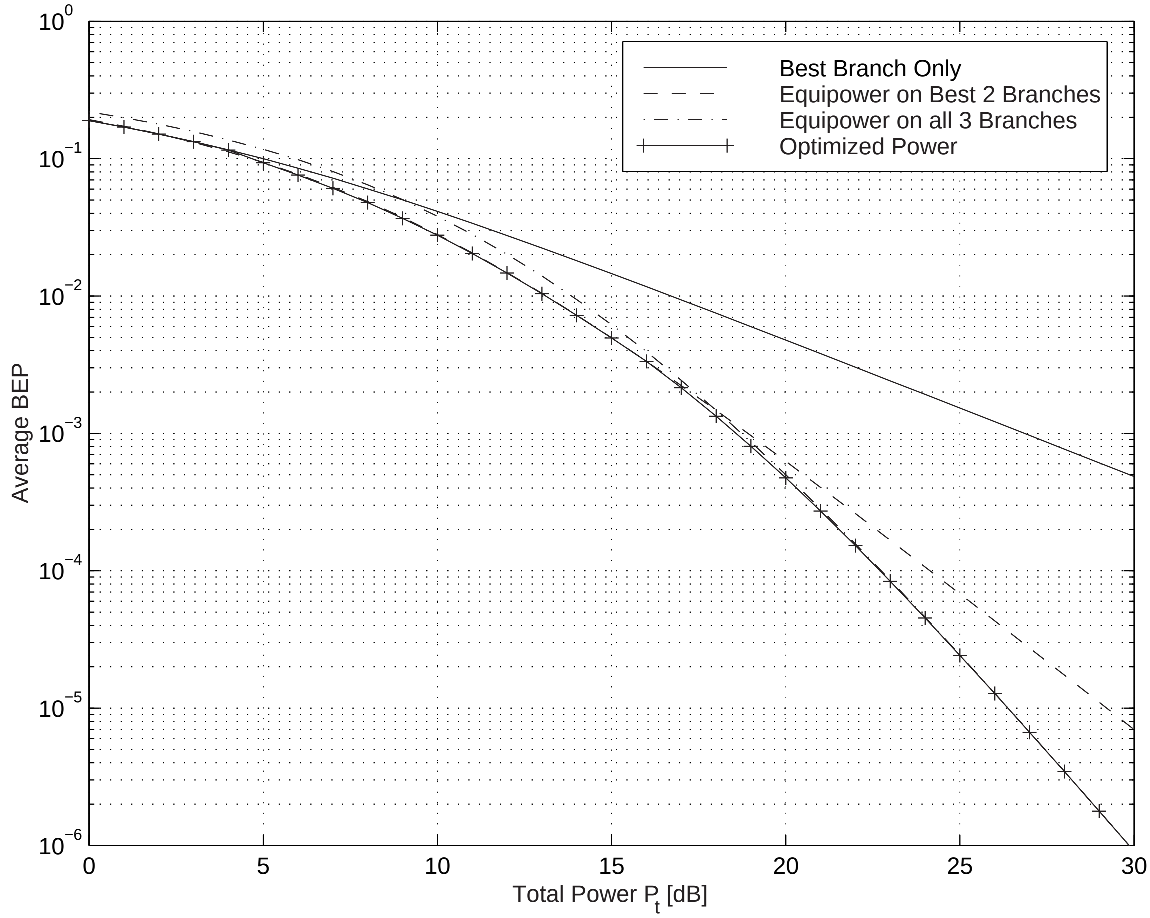
$$P_l = \text{Max}\left[\frac{P_t}{L} + \frac{1}{Lb}\sum_{k=1}^{L}\frac{1}{G_k} - \frac{1}{bG_l}, 0\right].$$

- Minimum average BER for 16-QAM

$$\begin{aligned} P_b(E) \;&=\; \frac{3}{4\pi}\int_0^{\frac{\pi}{2}}\prod_{l=1}^{L}\left(1 + \frac{2P_lG_l}{5m_l\sin^2\phi}\right)^{-m_l} d\phi \\ &+\; \frac{1}{4\pi}\int_0^{\frac{\pi}{2}}\prod_{l=1}^{L}\left(1 + \frac{18P_lG_l}{5m_l\sin^2\phi}\right)^{-m_l} d\phi. \end{aligned}$$

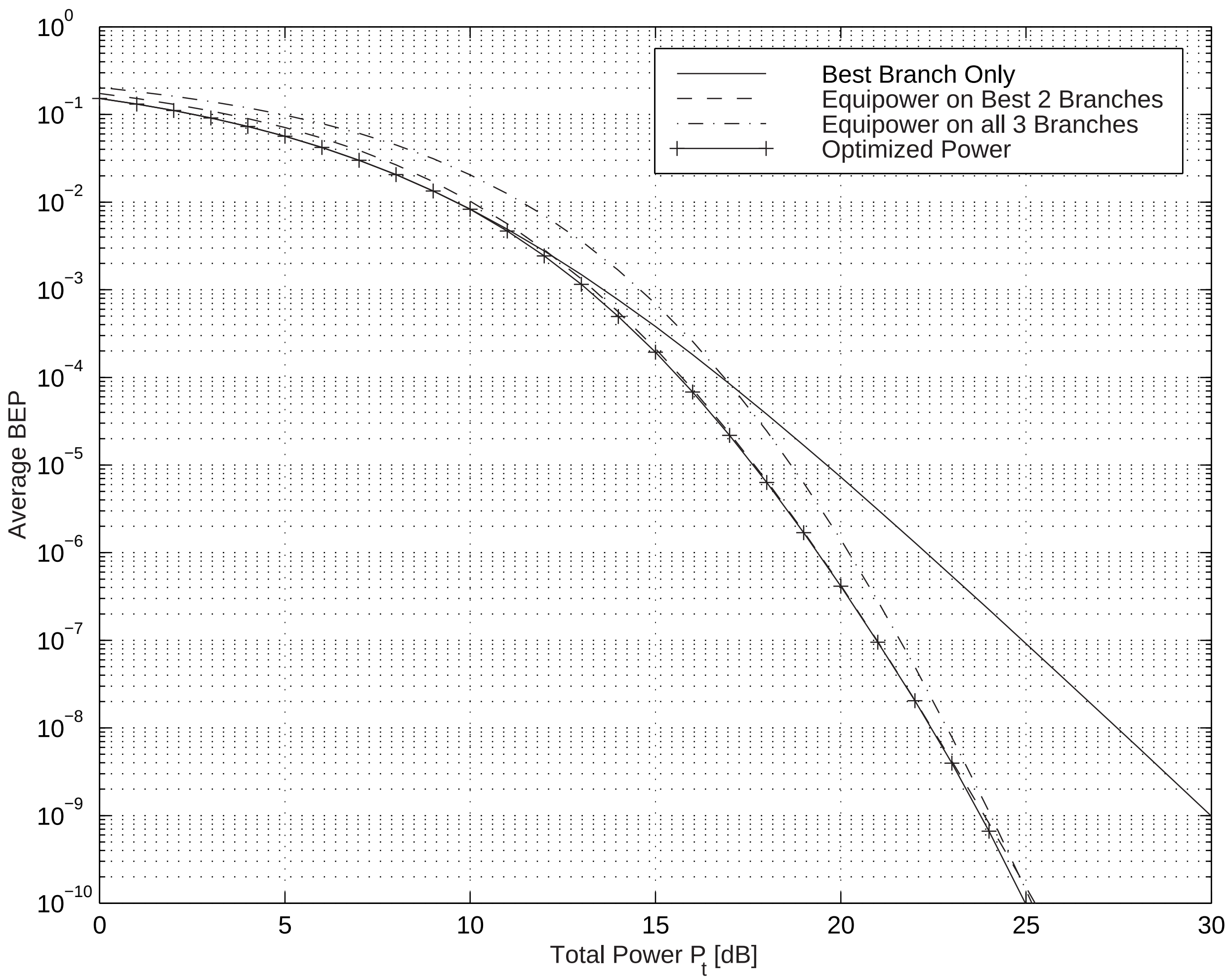
Rayleigh Fading

- $\Omega_2 = 0.5\,\Omega_1$ and $\Omega_3 = 0.1\,\Omega_1$



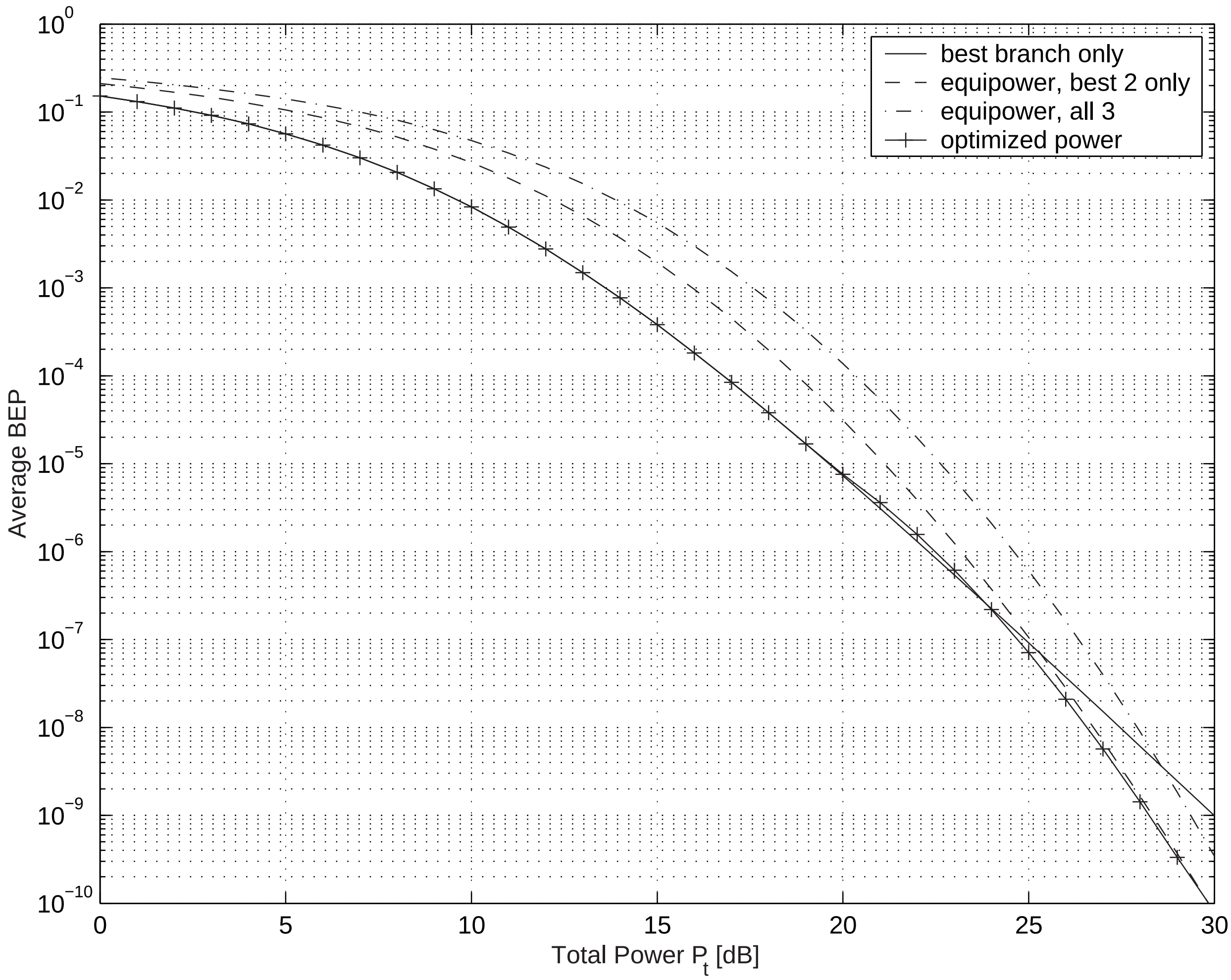
Nakagami Fading ($m = 4$)

- $\Omega_2 = 0.5\,\Omega_1$ and $\Omega_3 = \; 0.1\Omega_1$.



Nakagami Fading ($m = 4$)

- $\Omega_2 = 0.05\,\Omega_1$ and $\Omega_3 = 0.01\Omega_1$.



Model for MIMO Systems

- Consider a wireless link equipped with T antenna elements at the transmitter and R antenna elements at the receiver.
- The $R \times 1$ received vector at the receiver can be modeled as

$$\boldsymbol{r} = s_D \boldsymbol{H}_D \boldsymbol{w}_t + \boldsymbol{n},$$

where s_D is the transmitted signal of the desired user, $\boldsymbol{n}$ is the AWGN vector with zero mean and covariance matrix $\sigma_n^2 \boldsymbol{I}_R$, $\boldsymbol{w}_t$ represents the weight vector at the transmitter with $\|\boldsymbol{w_t}\|^2 = \Omega_D$, and $\boldsymbol{H}_D$ is the channel gain matrix for the desired user defined by

$$\boldsymbol{H}_D = \begin{pmatrix} h_{D,1,1} & h_{D,1,2} & \cdots & h_{D,1,T} \\ h_{D,2,1} & h_{D,2,2} & \cdots & h_{D,2,T} \\ \vdots & \vdots & \ddots & \vdots \\ h_{D,R,1} & h_{D,R,2} & \cdots & h_{D,R,T} \end{pmatrix}_{R \times T},$$

where $h_{D,i,j}$ denotes the complex channel gain for the desired user from the jth transmitter antenna element to the ith receiver antenna element.

MIMO MRC Systems

- Optimum combining vector at the receiver (given the transmitting weight vector $\boldsymbol{w}_t$) is

$$\boldsymbol{w}_r = \boldsymbol{H}_D \boldsymbol{w}_t.$$

- The resulting conditional (on $\boldsymbol{w}_t$) maximum SNR is

$$\mu = \frac{1}{\sigma_n^2} \boldsymbol{w}_t^H \boldsymbol{H}_D^H \boldsymbol{H}_D \boldsymbol{w}_t.$$

- Recall the **Rayleigh-Ritz Theorem**:
  For any non-zero $N \times 1$ complex vector $\boldsymbol{x}$ and a given $N \times N$ hermitian matrix $\boldsymbol{A}$,

$$0 < \boldsymbol{x}^H \boldsymbol{A} \boldsymbol{x} \leq \|\boldsymbol{x}\|^2 \lambda_{max},$$

  where λ_{max} is the largest eigenvalue of $\boldsymbol{A}$ and $\|\cdot\|$ denotes the norm. The equality holds if and only if $\boldsymbol{x}$ is along the direction of the eigenvector corresponding to λ_{max}.

- Apply Rayleigh-Ritz Theorem and use transmitting weight vector as

$$\boldsymbol{w}_t = \sqrt{\Omega_D} \boldsymbol{U}_{max},$$

  where $\boldsymbol{U}_{max}$ ($\|\boldsymbol{U}_{max}\| = 1$) denotes the eigenvector corresponding to the largest eigenvalue of the quadratic form

$$\boldsymbol{F} = \boldsymbol{H}_D^H \boldsymbol{H}_D.$$

MIMO MRC Systems (Continued)

- Maximum output SNR is given by

$$\mu = \frac{\Omega_D \sigma^2}{\sigma_n^2} \lambda_{max},$$

where λ_{max} is the largest eigenvalue of the matrix $\boldsymbol{H}_D^H \boldsymbol{H}_D$, or equivalently, the largest eigenvalue of $\boldsymbol{H}_D \boldsymbol{H}_D^H$.

- Outage probability below a target SNR μ_{th} in i.i.d. Rayleigh fading

$$\text{P}_{\text{out}} = \left| \boldsymbol{\Psi}_{\text{c}} \left(\frac{\sigma_n^2 \mu_{\text{th}}}{\Omega_D \sigma^2} \right) \right| \prod_{k=1}^{s} \frac{1}{\Gamma(t-k+1)\Gamma(s-k+1)}.$$

where $s = min(T, R)$, $t = max(T, R)$, and $\boldsymbol{\Psi}_{\text{c}}(x)$ is an $s \times s$ Hankel matrix function of $x \in (0, \infty)$ with entries given by

$$\{\boldsymbol{\Psi}_{\text{c}}(x)\}_{i,j} = \gamma(t - s + i + j - 1, x), \quad i, j = 1, \cdots, s.$$

and where $\gamma(\cdot, \cdot)$ is the incomplete gamma function.