Department of Telecomunications
Norwegian University of Science and Technology (NTNU)
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Problem Set

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Solutions: The detailed solution of all problems are available upon request from the instructor.

I- Review of Some Basics

Problem I.1: Log-Normal Distribution

Let $X_{\rm dB}$ be a normal random variable (RV) with mean $m_{X_{\rm dB}}$ and variance $\sigma^2_{X_{\rm dB}}$ and let $X = 10^{X_{\rm dB}/10}$ be the corresponding log-normal RV.
(a) Express the mean of X in terms of $m_{X_{\rm dB}}$ and $\sigma^2_{X_{\rm dB}}$.
(b) Express the variance of X in terms of $m_{X_{\rm dB}}$ and $\sigma^2_{X_{\rm dB}}$.
(c) Express the median of X in terms of $m_{X_{\rm dB}}$. Based on that, can you now explain why the area mean is sometimes referred to as the median link gain or median path loss.

Problem I.2: Outage Probability

Many wireless communication systems use the power outage probability as a performance measure, where the power outage probability is defined as the probability that the received power falls below some power threshold T_p. Typically, the bit error rate for received power below T_p is unacceptable for the desired application.

(a) Assume you received signal has a Rayleigh fading amplitude with an average fading power Ω.
(a-1) Derive the probability density function (PDF) of the fading power and deduce the power outage probability in terms of Ω and T_p ?
(a-2) Evaluate this outage probability for Ω= 20 dB and $T_p = 5$ dB.
(a-3) If your application requires a power outage probability of 10^{-2} for the threshold $T_p = 10$dB, what value of Ω is required ?

(b) Assume now that your received signal has a LOS component, so its amplitude has a Rician distribution with an average fading power Ω and a Rician factor K.
(b-1) Derive the PDF of the fading power then deduce the outage probability in terms of the Marcum Q-function[1], Ω, K, and T_p.
(b-2) Check that your answer reduces to the Rayleigh case (as given by (a-1)) for $K = 0$.
(b-3) What happens if K tends to infinity ?
(b-4) Plot the power outage probability as function of T_p/Ω (from -10 dB to 20 dB) for $K = 0$, $K = 5$ dB, and $K = 10$ dB. Use a dB scale on the x-axis and a log scale on the Y-axis. Comment on these curves.

[1] The Marcum Q-function is traditionally defined by $Q(a,b) = \int_b^\infty x \exp\left(-\frac{x^2+a^2}{2}\right) I_0(ax)\, dx$.

(c) Assume now that your received signal follows a Nakagami distribution with an average fading power Ω and a fading parameter m.
(c-1) Derive the PDF of the fading power then deduce the outage probability in terms of the complementary incomplete gamma function[2], Ω, m, and T_p. Show that you answer can be written in terms of a finite sum for the particular case when m is restricted to integer values.
(c-2) Check that your answer reduces to the Rayleigh case (as given by (a-1)) for $m = 1$.
(c-3) Plot the power outage probability as function of T_p/Ω (from -10 dB to 20 dB) for $m = 1$, $m = 2$, and $m = 4$. Use a dB scale on the x-axis and a log scale on the Y-axis. Comment on these curves.

Problem I.3: Average Outage Rate and Average Outage Duration

Consider a mobile operating in isotropic scattering conditions and without line of sight. We are interested in measuring the maximum average rate at which the faded signal envelope $\alpha(t)$ crosses a specified level R.

(a) Assuming that the local mean $\Omega = 3$ dB find the level R (in dB) that maximizes the average level crossing rate.

(b) Assuming that the mobile velocity is 50 Km/hr and the carrier frequency is 900 MHz, determine the average number of times the signal envelope will fade below the level found in (a) during a 1 minute test.

(c) How long, on average, will each fade in (b) last ?

Problem I.4: Average BER of BPSK over Rayleigh Fading

Show that the average BER of BPSK over Rayleigh fading channels is given by

$$P_b(E) = \frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}}{1+\overline{\gamma}}}\right).$$

Problem I.5: Performance of Noncoherent BFSK in a Nakagami Shadowed Environment

Consider binary orthogonal signaling using noncoherent FSK modulation and demodulation. The conditional bit error rate (BER) of noncoherent binary FSK is well know to be given by

$$P_b(E/\gamma) = \frac{1}{2}e^{-\gamma/2},$$

where $\gamma = \alpha^2 E_b/N_0$, α is the fading envelope, E_b is the energy-per-bit, and N_0 is the AWGN spectral density. Suppose that the signal is affected by flat Nakagami fading with fading parameter m and local mean $\Omega = E[\alpha^2]$.

(a) Determine the average BER $P_b(E)$ of noncoherent binary FSK over this channel in terms of m and the average SNR per bit $\overline{\gamma} = \Omega E_b/N_0$.

[2] The complementary incomplete gamma function is traditionally defined as $\Gamma(\alpha, x) = \int_x^\infty e^{-t}\, t^{\alpha-1}\, dt$.

(b) Assume that the local mean is subject to log-normal shadowing with area mean (logarithmic mean) μ_{dB} and shadowing standard deviation (logarithmic standard deviation) σ_{dB}.
(b-1) If an outage is declared when the average BER $P_b(E)$ (computed in (1)) exceeds a pre-determined threshold $\mathrm{BER_T}$, express the system outage probability in terms of m, μ_{dB}, σ_{dB}, E_b/N_0, and $\mathrm{BER_T}$.
(b-2) What is this outage probability for Rayleigh fading, $\mathrm{BER_T} = 10^{-3}$, $E_b/N_0 = 18$ dB, $\mu_{dB} = 25$ dB, and σ_{dB}=4 dB.

II- Diversity Systems

Problem II.1: Derivation of the Alternative Representation of the erfc Function

The erfc function is traditionally defined by

$$\text{erfc }(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt. \qquad (1)$$

In class, we have seen that the alternative representation of the erfc function is very useful when we want to evaluate the performance over fading channels. The alternative representation is :

$$\text{erfc }(x) = \frac{2}{\pi} \int_0^{\pi/2} e^{-x^2/\sin^2\theta} d\theta. \qquad (2)$$

(a) Starting with a product of two erfc functions, then going to polar coordinates, show the alternative representation of the erfc function. Using the same type of proof find an alternative desired form for the erfc ^{2} function.
(b) In the remainder of this problem, we shall derive the alternative representation of the erfc function by purely algebraic techniques.

1. Consider the integral

$$I_x(a) \triangleq \int_0^{\infty} \frac{e^{-at^2}}{x^2+t^2} dt. \qquad (3)$$

   Show that $I_x(a)$ satisfies the following differential equation :

$$x^2 I_x(a) - \frac{dI_x(a)}{da} = \frac{1}{2}\sqrt{\frac{\pi}{a}}. \qquad (4)$$

2. Solve the differential equation (**??**) and deduce that

$$I_x(a) = \frac{\pi}{2x} e^{ax^2} \text{erfc }(x\sqrt{a}). \qquad (5)$$

   `Hint`: $I_x(a)$ is a function in two variables x and a. However, since all our manipulations deal with a only, you can assume x to be a constant while solving the differential equation.

3. Setting $a = 1$ in (**??**) and making a suitable change of variables in the LHS of (**??**), derive the alternative representation of the erfc function :

$$\text{erfc }(x) = \frac{2}{\pi} \int_0^{\pi/2} e^{-x^2/\sin^2\theta} d\theta$$

Problem II.2: Performance of MRC Reception over Frequency-Selective Rayleigh Fading Channels

A spread-spectrum signal is transmitted over a frequency-selective fading channel constituted of L independent Rayleigh paths with an exponentially decaying power delay profile such that the lth path average fading power Ω_l is given by

$$\Omega_l = \Omega_1 e^{-\delta(l-1)}, \quad l = 1, \cdots L,$$

where $\delta \geq 0$ is the power decay factor and Ω_1 is the average fading power of the first path. A RAKE receiver resolves these L paths and combine them as per the rules of maximal ratio combining (MRC).

(a) Can you propose a combiner that, knowing the amplitude and phase of the fading on the L paths, improves upon the performance of MRC ?

(b) Assuming that MRC is used, express the average total SNR at the combiner output $\overline{\gamma}_t$ in terms of L, δ, and $\overline{\gamma}_1 = \Omega_1 E_b/N_0$.

(c) Assuming that a modulation scheme for which the conditional BER is given by

$$P_b(E/\gamma) = e^{-\gamma}$$

is used, find the average BER, $P_b(E)$, when $L = 3$, $\delta = 0.1$, and $\overline{\gamma}_t = 8$ dB.

Problem II.3: Performance of Switch-and-Stay Combining

Consider a dual branch ($L = 2$) switch and stay combiner (SSC). Let γ_{ssc} denote the SNR per bit at the output of the SSC combiner and let γ_T denote the predetermined switching threshold. Assume that the fading over the two branches is independent and identically distributed (i.i.d.) with an average SNR per bit per branch denoted by $\overline{\gamma}$.
(1) Show that the outage probability of SSC is given by

$$P_{out} = \begin{cases} P_\gamma(\gamma_T)\ P_\gamma(\gamma_{th}) & \gamma_{th} < \gamma_T \\ P_\gamma(\gamma_{th}) - P_\gamma(\gamma_T) + P_\gamma(\gamma_{th})\ P_\gamma(\gamma_T) & \gamma_{th} \geq \gamma_T, \end{cases}$$

where $P_\gamma(\gamma)$ is the CDF of the SNR per bit branch.
2 Consider the Nakagami-m fading case. Write an explicit closed-form expression for the outage probability of SSC in terms of the incomplete Gamma function. Plot on the same graph the outage probability of SSC (dashed lines) and SC (solid lines) as function of $\gamma_{th}/\overline{\gamma}$ (from -15 dB to 15 dB) and for m=1, m=2, and m=4. For SSC use $\gamma_T = \overline{\gamma}$. Compare with the outage probability without diversity reception and comment on your curves.
3 Find the PDF of the SNR at the SSC output in terms of the CDF $P_\gamma(\gamma)$ and the PDF $p_\gamma(\gamma)$ of the individual branches.
4 Consider again the Nakagami fading case.
4-1 Write an explicit closed-form expression for the PDF of the SNR at the SSC output.
4-2 Find a closed-form expression for the average bit error probability of SSC with DPSK over i.i.d. Nakagami fading channels.
4-3 Show that (for a fixed $\overline{\gamma}$ and m) there is an optimum value, in a minimum average bit error probability

sense, for the switching threshold γ_T.
4-4 Plot on the same graph the average probability of error of DPSK with dual-branch SSC (dashed lines) and dual-branch SC (solid lines) (you need to generalize the derivation done in class for the Rayleigh case to the Nakagami case) as function of $\overline{\gamma}$ (from 0 dB to 20 dB) and for m=1, m=2, and m=4. Compare with the DSPK curves without diversity and comment on your curves.

Problem II.4: Impact of Correlation on Macroscopic Dual-Branch Selection Combining

Consider a mobile in the soft handoff region. This mobile is continuously monitoring the local means of two pilot signals from two neighboring base stations and is being connected only to the base station with the strongest local mean. We assume that the two local means are log-normally distributed with the same area mean μ and the same shadowing standard deviation σ. Because of the common shadowing environment for the two base stations, we further assume that the two local means are correlated (i.e., the joint PDF of the two local mean dB values $10\log_{10}(\Omega_1)$ and $10\log_{10}(\Omega_2)$ follow a bivariate Gaussian distribution with correlation coefficient ρ).
(a) Derive a simple expression for the outage probability of this mobile and double check that you answer makes sense for $\rho = 0$ and $\rho = 1$. An expression which involves only one-fold finite-range integrals is acceptable. **Hint:** Show that the two-dimensional Gaussian Q-function defined by

$$Q(x_1, y_1; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{x_1}^{\infty}\int_{y_1}^{\infty} \exp\left\{-\frac{x^2+y^2-2\rho xy}{2\left(1-\rho^2\right)}\right\} dxdy$$

has the following desired representation

$$\begin{array}{l} Q(x_1, y_1; \rho) = \frac{1}{2\pi}\int_0^{\frac{\pi}{2}-\tan^{-1}\frac{y_1}{x_1}} \frac{\sqrt{1-\rho^2}}{1-\rho\sin 2\theta}\exp\left\{-\frac{x_1^2}{2}\frac{1-\rho\sin 2\theta}{(1-\rho^2)\sin^2\theta}\right\}d\theta \\ +\frac{1}{2\pi}\int_0^{\tan^{-1}\frac{y_1}{x_1}} \frac{\sqrt{1-\rho^2}}{1-\rho\sin 2\theta}\exp\left\{-\frac{y_1^2}{2}\frac{1-\rho\sin 2\theta}{(1-\rho^2)\sin^2\theta}\right\}d\theta \end{array} .x_1 \geq 0 y_1 \geq 0.$$

(b) Assume that $\mu = 10$ dB and $\sigma = 5$ dB. Plot the outage probability (use a log-scale for the Y-axis) as function of the outage threshold (use a dB scale for the X-axis) for $\rho = 0, 0.3, 0.6, 0.9$ and 1, then comment on the effect of the effect of shadowing correlation on the selection combining "macroscopic" diversity (as currently used in the soft handoff for IS-95).

Problem II.5: Impact of Correlation on Microscopic Dual-Branch Selection Combining

Assume that we are receiving via two antennas a signal that went through Rayleigh fading. We have one receiver which selects then detects only the signal with the highest SNR/bit/branch. We are going to assume that the average SNR/bit/branch of the antennas is the same, i.e., $\overline{\gamma}_1 = \overline{\gamma}_2 = \overline{\gamma}$. However, we are going to assume that these two antennas are at the mobile unit and that we do not have enough space to separate them by about half a wave-length. Hence, the fading over the two branches is going to be correlated. The objective of this problem is to quantify the effect of this correlation on the outage probability of the overall system. It is known that in this case the joint probability density function (PDF) of γ_1 and γ_2 is given by

$$p_{\gamma_1,\gamma_2}(\gamma_1,\gamma_2) = \frac{1}{\overline{\gamma}^2(1-\rho^2)}\, I_0\left(\frac{2\rho\sqrt{\gamma_1\gamma_2}}{\overline{\gamma}(1-\rho^2)}\right)\exp\left(-\frac{\gamma_1+\gamma_2}{\overline{\gamma}(1-\rho^2)}\right),\ \gamma_1 \geq 0, \gamma_2 \geq 0,$$

where $I_0(\cdot)$ is the 0-th order modified Bessel function of the first kind and ρ $(0 \leq \rho \leq 1)$ is a coefficient that quantifies the amount of correlation between the fading on the two branches, i.e., if $\rho = 0$ the fading over the two branches is uncorrelated and if $\rho = 1$ the fading on the two branches is fully correlated.
(a) Let $\gamma_{\max}$ denote the maximum of γ_1 and γ_2. Express the outage probability of $\gamma_{\max}$ (which is essentially the cumulative distribution function (CDF) of $\gamma_{\max}$ evaluated at a particular threshold γ_{th}) in terms of the complementary CDF of γ_1, complementary CDF of γ_2, and joint complementary CDF of γ_1 and γ_2.
(b) Show that the joint complementary CDF of γ_1 and γ_2 is given by

$$P[\gamma_1 > \gamma_{th}, \gamma_2 > \gamma_{th}] = 2e^{-\gamma_{th}/\overline{\gamma}} Q(\rho a, a) - e^{-a^2} I_0(\rho a^2),$$

where $a = \sqrt{2\gamma_{th}/(\overline{\gamma}(1-\rho^2))}$, and $Q(\cdot,\cdot)$ is the Marcum Q-function.
(c) Deduce that the outage probability of $\gamma_{\max}$ is given by

$$P_{out} = P[\gamma_{\max} \leq \gamma_{th}] = 1 - e^{-\gamma_{th}/\overline{\gamma}} \left(1 - Q(\rho a, a) + Q(a, \rho a)\right).$$

(d) Numerical problems can be encountered when the Marcum Q-function is programmed according to its traditional representation. Fortunately, this function has an alternative representation, which is well behaved and which is given by (you do not need to prove this result but if you would like to test your calculus skills you can do so)

$$Q(u,w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + \beta \sin\phi}{1 + 2\beta \sin\phi + \beta^2} \exp\left[-\frac{w^2}{2}\left(1 + 2\beta\sin\phi + \beta^2\right)\right] d\phi; \quad 0 \leq \beta = \frac{u}{w} < 1,$$

$$Q(u,w) = 1 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\beta^2 + \beta \sin\phi}{1 + 2\beta \sin\phi + \beta^2} \exp\left[-\frac{u^2}{2}\left(1 + 2\beta\sin\phi + \beta^2\right)\right] d\phi; \quad 0 \leq \beta = \frac{w}{u} < 1.$$

(d-1) In view of this alternative representation of the Marcum Q-function deduce that the outage probability is given by the following simple single finite-range integral representation

$$P_{out} = 1 - 2e^{-\gamma_{th}/\overline{\gamma}} + \frac{1-\rho^2}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-\frac{2\gamma_{th}}{\overline{\gamma}}\left(\frac{1+\rho\sin\phi}{1-\rho^2}\right)}}{1 + \rho^2 + 2\rho\sin\phi} \, d\phi.$$

(d-2) Show that this result makes sense for $\rho = 0$?
(e) Using the finite-range integral representation of the outage probability you are now in the position to compute numerically and plot P_{out} as function of $10\log_{10}\left(\frac{\gamma_{th}}{\overline{\gamma}}\right)$ for $\rho = 0, 0.2, 0.4, 0.6, 0.8, 0.9,$ and 0.99. Use a logarithmic scale on the Y-axis and a dB scale on the X-axis (going from -25 dB to 10 dB). Discuss the effect of correlation on the outage probability performance of this system.

Problem II.6: Impact of Correlation on Dual-Branch Maximal-Ratio Combining

Assume that we are receiving via two antennas a signal that went through Rayleigh fading channels. The signals are combined as per the rules of MRC. We are going to assume that the average SNR/bit/branch of the antennas is the same, i.e., $\overline{\gamma}_1 = \overline{\gamma}_2 = \overline{\gamma}$. However, we are going to assume that these two antennas are at the mobile unit and that we do not have enough space to separate them by about half a wave-length. Hence, the fading over the two branches is going to be correlated. The objective of this problem is to quantify the effect of this correlation on the performance of the system.
It is known that in this case the joint probability density function (PDF) of γ_1 and γ_2 is given by

$$p_{\gamma_1,\gamma_2}(\gamma_1,\gamma_2) = \frac{1}{\overline{\gamma}^2(1-\rho^2)} \, I_0\left(\frac{2\rho\sqrt{\gamma_1\gamma_2}}{\overline{\gamma}(1-\rho^2)}\right) \exp\left(-\frac{\gamma_1+\gamma_2}{\overline{\gamma}(1-\rho^2)}\right), \quad \gamma_1 \geq 0, \gamma_2 \geq 0,$$

where $I_0(\cdot)$ is the 0-th order modified Bessel function of the first kind and ρ ($0 \leq \rho \leq 1$) is a coefficient that quantifies the amount of correlation between the fading on the two branches, i.e., if $\rho = 0$ the fading over the two branches is uncorrelated and if $\rho = 1$ the fading on the two branches is fully correlated.
(a) Show that the MGF of the output (i.e., combined) SNR $\mathcal{M}_{\gamma_t}(s)$ is given by

$$\mathcal{M}_{\gamma_t}(s) = \left(1 - 2\overline{\gamma}\ s + (1-\rho^2)\overline{\gamma}^2\ s^2\right)^{-1};\quad s \leq 0.$$

(b) Show that fading correlation degrades the average symbol error probability of M-PSK when used in conjunction with dual-branch MRC. (You need to give a mathematical proof not just plots).
(c) Show that the PDF of the output SNR $p_{\gamma_t}(\gamma_t)$ is given by

$$p_{\gamma_t}(\gamma_t) = \frac{1}{2\rho\overline{\gamma}}\left[\exp\left[-\frac{\gamma_t}{(1+\rho)\overline{\gamma}}\right] - \exp\left[-\frac{\gamma_t}{(1-\rho)\overline{\gamma}}\right]\right];\quad \gamma_t \geq 0.$$

(d) Deduce a closed form expression for (i) the average BER of BPSK and (ii) the outage probability (i.e., the probability that the output SNR γ_t fall below a particular threshold γ_{th}. Double check that your answers make sense for $\rho = 0$ and $\rho = 1$.
5- (1 point) Plot the average BER of BPSK as function of the average SNR per bit per branch $\overline{\gamma}$ for $\rho = 0$, $\rho = 0.3$, $\rho = 0.6$, $\rho = 0.9$, and $\rho = 1$, and compare with the no-diversity case.

Problem II.7: Optimization of Transmit Diversity Systems

A signal is transmitted from a base station over L independent frequency diversity paths each of them being a slowly varying flat fading channel. At the receiver the mobile unit combines the multiple replicas as per the rules of maximal ratio combining. We assume that the transmitted signal on the lth carrier undergoes Nakagami-m flat fading with fading parameter m_l and average fading power $\Omega_l = E(\alpha_l^2)$ ($l = 1, \cdots, L$). As such the instantaneous signal-to-noise ratio (SNR) per symbol of the lth diversity channel is given by $\gamma_l = (\alpha_l^2 E_s^{(l)})/N_0 = (\alpha_l^2 P_l T_s)/N_0$, where N_0 is the AWGN power spectral density, α_l is the fading amplitude of the lth diversity path, $E_s^{(l)}$ is the energy per symbol over the lth diversity path, P_l is the power allocated to the lth carrier, and T_s is the symbol time. Denoting $G_l = (\Omega_l T_s)/N_0$, the average SNR of the lth path, $\bar{\gamma}_l$, can be written as $\bar{\gamma}_l = P_l G_l$.

We assume that the signal is transmitted over the L carriers using a modulation such us its conditional bit error rate (BER) (conditioned on the SNR γ) $P_b(E|\gamma)$ is well approximated by

$$P_b(E|\gamma) = a \cdot \exp(-b\gamma), \tag{6}$$

where a and b are constants. For example, a and b are 0.0852 and 0.4030, respectively, for the 16-QAM case that you will consider later on in our numerical examples.
(1) Show that the average BER is given by

$$P_b(E) = a \prod_{l=1}^{L}\left(1 + \frac{b\overline{\gamma_l}}{m_l}\right)^{-m_l}. \tag{7}$$

(b) Let P_t denote the total power which equals the sum of the powers P_l, i.e.,

$$P_t = \sum_{l=1}^{L} P_l. \tag{8}$$

on the L diversity paths.
(b-1) Show that there exists a unique set of powers $\{P_l\}_{l=1}^{L}$ which minimize the average BEP (**??**) subject to the total power constraint (**??**).
(b-2) Show that the optimum power allocation for minimum average BER is given by

$$P_l = m_l \ \text{Max}\left[\frac{P_t}{\sum_{k=1}^{L} m_k} + \frac{\sum_{k=1}^{L}\frac{m_k}{G_k}}{b\sum_{k=1}^{L} m_k} - \frac{1}{bG_l}, 0\right]. \tag{9}$$

Hint: You may want to use the Lagrange multiplier J given by

$$J = a\prod_{l=1}^{L}\left(1+\frac{bP_lG_l}{m_l}\right)^{-m_l} + \eta\left(\sum_{l=1}^{L} P_l - P_t\right).$$

and set all $\frac{\partial J}{\partial P_l} = 0 \ l = 1, \cdots, L$.
(c) Assume $L = 3$ and compare the average BER of 16-QAM with uniform (i.e. $P_1 = P_2 = P_3 = P_t/3$) and optimized power allocation over the 3 diversity paths. Assume that $m_1 = m_2 = m_3 = 4$, $\Omega_1 = 2\Omega_2 = 10\Omega_3$, and plot the average BER versus the total power P_t for the two power allocation strategies.

III- Co-Channel Interference

Problem III.1: Effect of Correlation on the Outage Probability

Consider the up-link in which a desired mobile is communicating with a base station (BS). This communication is subject to co-channel interference due to a single mobile in a neighboring cell. We assume that the local means (measured at the BS of interest) of the desired Ω_D and interfering Ω_I mobiles are log-normally distributed with area means (logarithmic means) $\mu_{D_{dB}}$ and $\mu_{I_{dB}}$, respectively, and with the same shadowing standard deviation (logarithmic standard deviation) $\sigma_{\Omega_{dB}}$. Because of the common shadowing environment that the desired and interfering signals undergo in the neighborhood of the BS of interest, we further assume that these two local means are correlated (i.e., the joint PDF of the two local mean dB values $10\log_{10}\Omega_D$ and $10\log_{10}\Omega_I$ follow a bivariate Gaussian distribution with correlation coefficient ρ.

(a) Assuming that an outage event is declared if the carrier-to-interference ratio (CIR) $\Lambda = \Omega_D/\Omega_I$ falls below a predetermined threshold Λ_{th}, express the outage probability of the up-link under consideration in terms of Λ_{th}, $\mu_{D_{dB}}$, $\mu_{I_{dB}}$, $\sigma_{\Omega_{dB}}$, and ρ.

(b) Deduce the effect of the correlation between the desired and interfering signal on the up-link performance (i.e., explain if this correlation improve or degrade the up-link performance).

Problem III.2: Outage Probability of Fully-Loaded Systems

Consider an FDMA macro-cellular mobile radio system in which the desired signal power s_d is independent from the N_I interfering signal powers $\{s_i\}_{i=1}^{N_I}$. The desired user has an average fading power (i.e., local mean) denoted by $\overline{s}_d$ and is subject to slowly-varying flat Rayleigh fading. The N_I active interfering signals are assumed to be independent, to have the same average fading power denoted by $\overline{s}_i$, and to be subject to slowly-varying flat Rayleigh fading.
(a) Find the probability density function (PDF) of the total interference power $s_I = \sum_{n=1}^{N_I} s_i$.

(b) Show that for interference limited systems the outage probability $P_{\text{out}} = \Pr[s_d/s_I \leq \lambda_{th}]$ is given by the following compact closed-form expression

$$P_{\text{out}} = 1 - \left(1 + \frac{\lambda_{th}\ \overline{s}_i}{\overline{s}_d}\right)^{-N_I}$$

(c) Assuming a fully-loaded system ($N_I = 6$), a protection ratio $\lambda_{th} = 18$ dB, and an average fading power for the interferers $\overline{s}_i = 2$ dB, what should be the average power of the desired signal $\overline{s}_d$ (in dB) to meet an outage probability requirement of $P_{\text{out}} = 10^{-2}$.

Problem III.3: Outage Probability of Partially-Loaded Systems

Consider an FDMA macro-cellular mobile radio system in which the cells are divided into 60 degrees sectors and each cell has N_s available voice channels. The desired signal power s_d is independent from the interfering signal powers s_i. The desired user has an average fading power (i.e., local mean) denoted by $\overline{s}_d$ and is subject to slowly-varying flat Rayleigh fading. The active interfering signals are assumed to have the same average fading power denoted by $\overline{s}_i$, and to be subject to slowly-varying flat Rayleigh fading.

An outage is declared if either the carrier-to-interference-ratio (CIR) falls below a predetermined threshold λ_{th} or the desired signal power falls below another specified threshold s_{th}. Assuming that the system was designed for a blocking probability B, derive a closed-form expression for the outage probability of this system (in terms of B, N_s, $\overline{s}_d$, $\overline{s}_i$, λ_{th}, and s_{th}).

Problem III.4: Outage Probability with Non IID Interferers

Consider a cellular system with N_I co-channel interferers.
(1) Assume that the desired user is Nakagami distributed with fading parameter m_d and local mean Ω_d and that the interferers are i.i.d. Nakagami with fading parameters $m_I = m_d = m$ and local mean Ω_I.
(1-a) Find the distribution of the total interference power $s_I = \sum_{k=1}^{N_I} s_k$.
(1-b) Show that for interference limited systems

$$P_{out} = P_r(\lambda = \frac{s_d}{s_I} \leq \lambda_{th}) = I_x(m, mN_I)$$

where $x = \frac{1}{1+\frac{\Omega_d}{\Omega_I \lambda_{th}}}$ and $I_x(\cdot,\cdot)$ is the incomplete Beta function ratio as defined in class.
(1-c) Find the outage probability when we also impose a minimum desired signal power requirement i.e. $P_{out} = P_r(\lambda \leq \lambda_{th} \text{ or } s_d \leq s_{th})$. Express your answer in terms of the incomplete Gamma function.
(1-d) Plot the outage probability of **(1-b)** and **(1-c)** as function of the normalized "average" $SIR = \frac{\Omega_d}{\Omega_I \lambda_{th}}$ for $m = 2$ and $N_I = 6$. For **(1-c)** use $s_{th} = 17$ dB and $\Omega_d = 40$ dB. Use log scale for the y-axis and dB scale (from 0 to 30 dB) for the X-axis.
(2) Assume that the desired user is Nakagami distributed with fading parameter m_d and local mean Ω_d and that the interferers are not necessarily i.i.d. Nakagami with fading parameter m_k, $k = 1, 2, \ldots, N_I$.
(2-a) Assume local mean power Ω_k, $k = 1, 2, \ldots, N_I$. Applying the Gil-Palaez lemma covered in class, show that the outage probability (in the case of interference limited systems) is given by

$$P_{out} = P_r\left(\lambda = \frac{s_d}{\sum_{k=1}^{N_I} s_k} \leq \lambda_{th}\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \frac{\sin\left(\sum_{k=1}^{N_I} m_k \tan^{-1}\left(\frac{\lambda_{th}\Omega_k}{m_k}t\right) - m_d \tan^{-1}\left(\frac{\Omega_d}{m_d}t\right)\right)}{t\left(1+\left(\frac{\Omega_d}{m_d}t\right)^2\right)^{\frac{m_d}{2}} \prod_{k=1}^{N_I}\left(1+\left(\frac{\lambda_{th}\Omega_k}{m_k}t\right)^2\right)^{\frac{m_k}{2}}}\, dt$$

(2-b) Let $\lambda_{th} = 18$ dB, $m_d = 2$, $N_I = 3$, $m_1 = 1, m_2 = 0.5, m_3 = 0.75$, and $\Omega_1 = 5$ dB, $\Omega_2 = 2$ dB, $\Omega_3 = 1$ dB. Plot P_{out} as function of Ω_d (from 0 to 20 dB).

Problem III.5: Diversity to Combat Co-Channel Interference

Diversity can also be used to improve the performance of wireless systems subject to co-channel interference. Consider a dual-branch diversity receiver in the presence of a single co-channel interferer. We will compare the outage probability of this system (assuming a minimum desired signal power constraint s_{th}) when selection combining or switching combining are used.

(1) Assume that both the desired and interfering signals independent and are both subject to Rayleigh type of fading with local mean Ω_D and Ω_I, respectively.
(1-a) Derive the outage probability of a SC diversity system in which the receiver picks and processes only the branch with the best desired signal.
(1-b) Deduce the outage probability formulas for the limiting (interference-limited and noise-limited) cases.
(2) Assume again that the desired and interfering signals are independent and are both subject to Rayleigh type of fading with local mean Ω_D and Ω_I, respectively.
(2-a) Derive the outage probability of a SSC diversity system in which the receiver switches as per the rules of SSC according to the variations of the desired signal power.
(2-b) Find the optimal switching threshold that minimizes the outage probability.
(2-c) Deduce the outage probability and optimal switching threshold formulas for the limiting (interference-limited and noise-limited) cases.
(3) Illustrate the diversity gain by plotting the outage probability as function of the normalized CIR ($\Omega_D/(\Omega_I \lambda_{th})$) for (1) without diversity, (2) SSC diversity with optimal threshold, and (3) SC diversity. Assume $\Omega_D/s_{th} = 10$ dB and use a log-scale on the Y-axis. Comment on your curves.
(4) Actually in presence of co-channel interference, the combining decision algorithm is not unique. For example, with SC the selection of the branch can be based on total (desired plus interference) power, CIR, or desired signal power (as analyzed in (1)). Since the two former decisions algorithms are harder to analyze in closed-form, evaluate their outage probability performance by using Monte-Carlo simulations. In your sumulations, consider SC only and plot the outage probability as function of the normalized CIR ($\Omega_D/(\Omega_I \lambda_{th})$) for the three different decision algorithms. Again assume $\Omega_D/s_{th} = 10$ dB and use a log-scale on the Y-axis. Comment on your curves.

Part IV- Multi-hop Communication Systems

Problem IV.1: End-to-End Average BER of Dual-Hop Communication Systems

Consider a dual-hop wireless communication system in which two terminals are communicating via a third terminal that acts as a relay. As such the signal propagates from the source terminal to the destination terminal through the two hops/links in series. Assume that the two hops are subject to independent Rayleigh fading with average SNR per bit $\overline{\gamma}_1$ and $\overline{\gamma}_2$, respectively. Binary DPSK for which the conditional BER is given by

$$P_b(E/\gamma) = \frac{1}{2}e^{-\gamma}$$

is used on the two links. Assume that the relay acts in a regenerative fashion (i.e., decodes the received signal from the source and then transmits the detected version to the destination terminal).

(a) Show that the end-to-end (i.e. from source to destination) average BER is given by

$$P_b(E) = \frac{1 + \overline{\gamma}_1 + \overline{\gamma}_2}{2\,(1 + \overline{\gamma}_1)(1 + \overline{\gamma}_2)}.$$

(b) What happens to the end-to-end average BER if $\overline{\gamma}_1 = \overline{\gamma}_2 = \overline{\gamma}$ and $\overline{\gamma} << 1$ or $\overline{\gamma} >> 1$?

Problem IV.2: Outage Probability of Multi-hop Communication Systems over Log-Normal Shadowed Channels

"Ad-Hoc" networks rely on the idea of taking advantage of the mobile users themselves to act as nodes of the network. For instance, in these kind of networks, there is no need for an infrastructure of base stations to carry information between mobile users. Users that are close to each other communicate directly while users that are far away communicate via other users that act as "relays". There are many issues related to this topic, but we will focus in this problem on the end-end outage probability of these "multi-hop" links where the information is conveyed from the transmitting user to the receiving user via multiple relay users.

Consider a multi-hop wireless system consisting of N hops. The source user is transmitting the information to a destination user via $N-1$ users that act as relays of the information. The N hops are subject to log-normal shadowing and are assumed to be independent and identically distributed (i.i.d). As such the carrier-to-noise ratio (CNR) of the N hops $\{\Gamma_n\}_{n=1}^{N}$ are independent and log-normally distributed with the same logarithmic mean μ and the same shadow logarithmic standard deviation σ (i.e., $10\log_{10}\Gamma_n$ are Gaussian with mean μ and standard deviation σ for all $n = 1, \cdots N$.). In this problem, we are going to compare two strategies of relaying the information.

(a) In the first strategy, termed the "decode-and-forward" or the "regenerative" strategy, the relay detects and decodes the signal, regenerates the symbols, then re-transmits them to the next relay or the destination user. An outage event is declared if the CNR of any of the N hops falls below an acceptable predetermined threshold Γ_{th}. Derive a simple formula for the end-to-end (from the source to the destination) outage probability P_{out}^{df} for this first strategy. Express your answer in terms of μ, σ, N, and Γ_{th}. (**Hint:** Think about the probability of no outage).
(b) In the second strategy, termed the "amplify-and-forward" or the "non-regenerative" strategy, the relay does not attempt to detect or decode the signal. It just amplifies it and re-transmits it to the next relay or the destination user. It can be shown that the end-to-end CNR Γ_{af} of a system using this strategy is given (under certain conditions that are beyond the scope of this problem) by

$$\frac{1}{\Gamma_{af}} = \frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \cdots + \frac{1}{\Gamma_N}. \tag{10}$$

An outage event is declared if the end-to-end CNR Γ_{af} falls below an acceptable predetermined threshold Γ_{th}. Express the end-to-end outage probability P_{out}^{af} in terms of μ, σ, N, and Γ_{th}. (**Hint:** You may want to rely on the Fenton-Wilkinson method/approximation for finding the statistics of the sum of log-normal random variables (see for example pages 129-131 from Stuber textbook, 2nd Edition)).
(c) Focus on the two-hop case ($N = 2$), fix $\mu = 10$ and $\sigma = 4$ and plot on the same figure the end-to-end outage probability for both strategies as function of the outage threshold Γ_{th}. Use a log-scale for the end-to-end outage probability and a dB scale for the outage threshold. Compare and comment on the

end-to-end outage performance of the two strategies.
(d) Fix $\mu = 10$, $\sigma = 4$, and $\Gamma_{th} = 1$ (0 dB) and plot on the same figure the end-to-end outage probability (on a log-scale) for both strategies as function of the number of hops N. Comment on and compare the end-to-end outage performance behavior with respect to the number of hops for both strategies.