Voting Almost Maximizes Social Welfare Despite Limited Communication*

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Abstract

In cooperative multiagent systems an alternative that maximizes the *social welfare*—the sum of utilities—can only be selected if each agent reports its full utility function. This may be infeasible in environments where communication is restricted. Employing a voting rule to choose an alternative greatly reduces the communication burden, but leads to a possible gap between the social welfare of the optimal alternative and the social welfare of the one that is ultimately elected. Procaccia and Rosenschein [11] have introduced the concept of *distortion* to quantify this gap.

In this paper, we present the notion of *embeddings into voting rules*: functions that receive an agent's utility function and return the agent's vote. We establish that very low distortion can be obtained using randomized embeddings, especially when the number of agents is large compared to the number of alternatives. We investigate our ideas in the context of three prominent voting rules with low communication costs: Plurality, Approval, and Veto. Our results arguably provide a compelling reason for employing voting in cooperative multiagent systems.

1 Introduction

A major challenge that arises in the design and implementation of multiagent systems is the aggregation of the preferences of the agents. Voting theory provides a neat solution by giving extremely well-studied methods of preference aggregation. In recent years the theoretical aspects of computational voting have been enthusiastically investigated, especially within the AI community (see, e.g., [13, Chapter 1] and the many references therein). Moreover, voting has been applied for preference aggregation in areas as diverse as Planning, Scheduling, Recommender Systems, Collaborative Filtering, Information Extraction, and Computational Linguistics (see, e.g., [5, 10, 14]).

While the appeal of voting in the context of heterogeneous, competitive multiagent systems is apparent, some multiagent systems are *centrally designed* and *fully cooperative* (e.g., systems for planning and scheduling, recommender systems, collaborative filtering, and so on). We believe that, to date, the benefit of employing voting in such domains was unclear. Indeed, agents are normally assumed to compute a utility for every possible alternative. If the agents are cooperative then they can simply communicate their utilities for the different alternatives, and subsequently select an alternative that maximizes the *social welfare*, i.e., the sum of utilities.

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However, accurately conveying an agent's utility function for each alternative may be very costly in terms of communication. This could prove to be a serious obstacle in domains where communication is restricted. Communication may be limited by the physical properties of the system (e.g., slow or error-prone transmitters, systems with low energy consumption requirements, etc.) or the representation of the full utility functions may require a huge amount of information. Blumrosen et al. [1] outline additional persuasive reasons why communication should be restricted in multiagent settings. Fortunately, some prominent voting rules—functions that select an alternative given the preferences of the agents—impose a very small communication burden [4], and are moreover resistant to errors in communication [12].

For example, consider the paradigmatic cooperative multiagent system domain: scanning an area on Mars with multiple rovers (which are known to have limited communication capabilities). Suppose the rovers must select or update their joint plan (this may happen very often), and there are one million alternatives. Moreover, suppose each rover computes a utility for each alternative on a scale of one to one million (this is, in fact, a very coarse scale). A rover would need to communicate $10^6 \cdot \log(10^6) \approx 20 \cdot 10^6$ bits in order to report its utility function. In contrast, under the *Plurality* voting rule, where each agent votes for a single alternative and the alternative with most votes wins, a rover only needs to transmit twenty bits. Even though current applications may involve a small number of rovers, the research in wireless communication systems already envisages large-scale applications (e.g., for environment monitoring, disaster relief, battlefield operations and surveillance) with many ultra-small, possibly mobile, wireless devices such as sensors or mini-robots that cooperate towards a common goal. Such devices are expected to be fully autonomous, a property that calls for low energy consumption and, consequently, for low communication requirements. The Harvard Micro Air Vehicles Project[1] provides a concrete example of such a system.

In this paper we shall argue that, in some cooperative multiagent systems, exact maximization of the social welfare can be replaced by very simple voting rules (given an extra ingredient that we present below). The benefit is a huge reduction in the communication burden, whereas the cost, a deterioration in the social welfare of the outcome, will be shown to be almost negligible in some settings. This arguably provides a pivotal reason for employing voting in cooperative multiagent systems, and in AI in general.

Our approach. The degree to which the social welfare of the outcome can decrease when voting is used is captured by the notion of *distortion*, introduced by Procaccia and Rosenschein [11]. They focus on voting rules that receive as input a ranking of the alternatives, and, crucially, assume that each agent reports a ranking such that the alternative that is ranked in the k'th place has the k'th highest utility. Under this assumption, they define the distortion of a voting rule to be the worst-case ratio between the maximum social welfare over all the alternatives, and the social welfare of the winner of the election; the worst-case is taken over all the possible utility functions of the agents. After proving some impossibility results, Procaccia and Rosenschein further restrict the structure of the utility functions. Even under this additional (very strong) assumption, they show that the distortion of most prominent voting rules is linear in the number of alternatives. The approach of Procaccia and Rosenschein is *descriptive*: they propose to use the notion of distortion as a criterion in the comparison of different voting rules.

Our main conceptual contribution is the consideration of *embeddings into voting rules*. An embedding is a set of instructions that informs each agent how to vote, based only on the agent's own utility function, that is, without any communication or coordination between different agents.

[1] http://robobees.seas.harvard.edu

More accurately, an embedding into a specific voting rule is a function from utility functions to votes that are valid under the voting rule. For instance, consider the simple Plurality rule described above. Given a utility function, an embedding into Plurality returns the alternative that the agent votes for. Procaccia and Rosenschein implicitly use one specific embedding, but many different embeddings exist. In this sense, our approach is *algorithmic*: we wish to *design* embeddings in a way that minimizes the distortion.

We redefine the notion of distortion to take embeddings into account. The *distortion of an embedding into a voting rule* is still the worst-case ratio between the maximum social welfare and the social welfare of the winner, but now the winner depends both on the voting rule and on the embedding, that is, on the way the utilities of the agents are translated into votes. The worst-case is taken over all possible utilities; we do not make any assumption regarding the utilities, except that they are normalized.

We take the idea of embeddings into voting rules one step further by allowing *randomized embeddings.* A randomized embedding randomly chooses the agent's vote, according to some probability distribution. The distortion is defined similarly, by taking into account the *expected* social welfare of the winner of the election. As we shall see, randomization gives us great power and flexibility, and ultimately provides us with the tools to design truly low-distortion embeddings.

We wish to design low distortion embeddings into voting rules with low communication complexity. Indeed, given that each of our cooperative agents votes according to the instructions provided by the embedding (in a fully *decentralized* way), then an alternative with social welfare close to optimal may be elected in the face of restricted communication. We find the existence of low distortion embeddings rather striking, as the social welfare is a *centralized* concept.

Our results. We study the distortion of embeddings into three voting rules: Plurality, Approval (each agent approves a subset of alternatives), and Veto (each agent gives a "negative point" to one alternative). Plurality and Veto have the smallest communication burden among all prominent voting rules: only $\log m$ bits per agent, where m is the number of alternatives. Approval requires more communication, m bits per agent, but still less than other prominent voting rules.

We first deal with the Plurality rule. We show that any deterministic embedding into Plurality has distortion $\Omega(m^2)$, and also provide a matching upper bound. Our main result deals with randomized embeddings into Plurality: we show that the naïve embedding into Plurality, which selects an alternative with probability proportional to its utility, yields constant distortion when $n = \Omega(m \ln m)$, where n is the number of agents, and has extremely low distortion, specifically $1 + o(1)$, for larger values of n.

Next we investigate the Approval rule. We give a lower bound of $\Omega(m)$ for deterministic embeddings, and also present a matching upper bound. Our randomized upper bounds for Approval follow directly from the upper bounds for Plurality, since any embedding into Plurality is also an embedding into Approval.

These results apply to the case $n \geq m$. Even though we have no positive theoretical results for the case $n \leq m$, we present experimental results from the application of randomized embeddings into Plurality and Approval on random utility profiles; the results suggest that relatively low distortion can also be achieved in this case as well provided that the number of agents is not very small.

Finally, we consider the Veto rule. We show that any deterministic embedding into Veto has infinite distortion, and the same is true for randomized embeddings if $n < m - 1$. We further show that low-distortion embeddings into Veto can be obtained, albeit using a large number of agents. Our related positive result is stated in a more general form and applies to *any* scoring protocol.

2 Embeddings into Plurality

We denote by $N = \{1, \ldots, n\}$ the set of *agents*, and by A, $|A| = m$, the set of *alternatives*.

We assume that the agents have normalized cardinal utilities over A. Specifically, let $\mathcal{U} = \mathcal{U}(A)$ be the set of utility functions u over A such that for each $x \in A$, $u(x) \geq 0$, and $\sum_{x \in A} u(x) = 1$. Each agent i has a utility function $u \in \mathcal{U}$. A *utility profile* is a vector of utility functions

$$\mathbf{u} = \langle u_1, \ldots, u_n \rangle \in \mathcal{U}^n \quad .$$

The *social welfare* of an alternative $x \in A$ with respect to $\mathbf{u} \in \mathcal{U}^n$, denoted $\text{sw}(x, \mathbf{u})$, is the sum of the utilities of x for all agents:

$$\text{sw}(x, \mathbf{u}) = \sum_{i \in N} u_i(x) \quad .$$

In our formal presentation, a voting rule is defined as a function that selects a *set* of alternatives rather than a single alternative. Such a function is formally known as a *voting correspondence*, hence the term *voting rule* is slightly abused. We must deal with sets of winners since our rules are based on notions of score, and there might be a tie with respect to the maximum score.

Under the *Plurality* rule, each agent casts its vote in favor of a single alternative. The set of winners is the set of alternatives with a maximum number of votes.

A *deterministic embedding into Plurality* is a function $f : \mathcal{U} \to A$. Informally, given an agent $i \in N$ with a utility function $u \in \mathcal{U}$, $f(u)$ is the alternative which agent i votes for under the embedding f. Given a utility profile $\mathbf{u} \in \mathcal{U}^n$ and an embedding f, denote the (Plurality) *score* of $x \in A$ by

$$\text{sc}(x, f, \mathbf{u}) = |\{i \in N \ : \ f(u_i) = x\}| \quad ,$$

and denote the set of *winners* by

$$\text{win}(f, \mathbf{u}) = \text{argmax}_{x \in A} \text{sc}(x, f, \mathbf{u}) \quad .$$

Note that the argmax function returns a *set* of maximal alternatives.

A *randomized embedding* randomly selects one of the alternatives, that is, it is a function $f : \mathcal{U} \to \Delta(A)$, where $\Delta(A)$ is the space of probability distributions over A. Put another way, given $u \in \mathcal{U}$, $f(u)$ is a random variable that takes the value $x \in A$ with probability $p(x)$, i.e., $\sum_{x \in A} p(x) = 1$. With respect to a randomized embedding f, $\text{sc}(x, f, \mathbf{u})$ is a random variable that takes values in $\{1, \ldots, n\}$, and $\text{win}(f, \mathbf{u})$ is a random variable that takes values in 2^A, the powerset of A. Less formally, given a randomized embedding f, a utility profile $\mathbf{u}$, and $S \subseteq A$, we have some probability (possibly zero) of S being the set of winners when f is applied to $\mathbf{u}$.

As a measure of the quality of an embedding, we use the notion of distortion, introduced by Procaccia and Rosenschein [11], but adapt it to apply to general embeddings.

Definition 1 (Distortion)**.**

1. Let $f : \mathcal{U} \to A$ be a deterministic embedding, $\mathbf{u} \in \mathcal{U}^n$. The *distortion of f at* $\mathbf{u}$ is

$$\text{dist}(f, \mathbf{u}) = \frac{\max_{y \in A} \text{sw}(y, \mathbf{u})}{\min_{x \in \text{win}(f, \mathbf{u})} \text{sw}(x, \mathbf{u})} \quad .$$

2. Let $f : \mathcal{U} \to \Delta(A)$ be a randomized embedding, $\mathbf{u} \in \mathcal{U}^n$. The *distortion of f at $\mathbf{u}$* is

$$\mathrm{dist}(f, \mathbf{u}) = \frac{\max_{y \in A} \mathrm{sw}(y, \mathbf{u})}{\mathbb{E}\left[\min_{x \in \mathrm{win}(f,\mathbf{u})} \mathrm{sw}(x, \mathbf{u})\right]} \quad .$$

3. Let f be a deterministic or randomized embedding. The *distortion of f* is

$$\mathrm{dist}(f) = \max_{\mathbf{u} \in \mathcal{U}^n} \mathrm{dist}(f, \mathbf{u}) \quad .$$

Let us give an intuitive interpretation of this important definition. The distortion of a deterministic embedding is the worst-case ratio between the social welfare of the most popular alternative, and the social welfare of the least popular winner, where the worst-case is with respect to all possible utility profiles. In other words, we are interested in the question: how small can the social welfare of one of the winners be, when compared to the alternative with maximum social welfare?

Our focus on the social welfare of the "worst" winner is appropriate since the analysis is worst-case. Alternatively, it is possible to think of voting rules that elect only one of the alternatives with maximum score, but in the worst-case the most unpopular one is elected, that is, in the worst-case ties are broken in favor of alternatives with lower social welfare; this is the interpretation of Procaccia and Rosenschein [11].

The definition of distortion with respect to randomized embeddings is slightly more subtle. Here there is no definite winner. However, given a utility profile $\mathbf{u} \in \mathcal{U}$, we can talk about the expected minimum social welfare among the winners, since the set of winners is simply a random variable that takes values in 2^A, hence $\min_{x \in \mathrm{win}(f,\mathbf{u})} \mathrm{sw}(x, \mathbf{u})$ is a random variable that takes values in the interval $[0, n]$ and its expectation is well-defined. The rest of the definition is identical to the deterministic case.

2.1 Deterministic Embeddings

Procaccia and Rosenschein [11] consider a specific, naïve deterministic embedding into Plurality. Their embedding simply maps a utility function $u \in \mathcal{U}$ to an alternative with maximum utility, that is, $f(u) \in \mathrm{argmax}_{x \in A} u(x)$. They show that its distortion is $m - 1$ under a very restricted definition of distortion (called *misrepresentation*) that assumes a specific structure of utility functions.

It is easy to see that the distortion of this naïve embedding, according to Definition 1, is at most m^2. Indeed, let $\mathbf{u} \in \mathcal{U}$, and let $x \in \mathrm{win}(f, \mathbf{u})$, where f is the naïve embedding. By the Pigeonhole Principle, it must hold that $\mathrm{sc}(x, f, \mathbf{u}) \geq n/m$. Now, for each agent $i \in N$ such that $f(u_i) = x$, it must hold that $u_i(x) \geq 1/m$, since x has maximum utility and there must exist an alternative with utility $1/m$ (again, by the Pigeonhole principle). We deduce that $\mathrm{sw}(x, \mathbf{u}) \geq n/m^2$. On the other hand, for any $y \in A$, $\mathrm{sw}(y, \mathbf{u}) \leq n$. Therefore,

$$\mathrm{dist}(f) = \max_{\mathbf{u} \in \mathcal{U}^n} \frac{\max_{y \in A} \mathrm{sw}(y, \mathbf{u})}{\min_{x \in \mathrm{win}(f,\mathbf{u})} \mathrm{sw}(x, \mathbf{u})} \leq \frac{n}{n/m^2} = m^2 \quad .$$

We wish to ask whether there is a clever deterministic embedding into Plurality that (asymptotically) beats the m^2 upper bound given by the naïve one. Our first theorem answers this question in the negative.

Theorem 2. *Let $|A| = m \geq 3$, $|N| = n \geq \left\lceil \frac{m+1}{2} \right\rceil$, and let $f : \mathcal{U} \to A$ be a deterministic embedding into Plurality. Then dist$(f) = \Omega(m^2)$.*

Proof. Let f be a deterministic embedding into Plurality. For every pair of distinct alternatives $x, y \in A$, let $u^{xy} \in \mathcal{U}$ such that $u^{xy}(x) = 1/2$, $u^{xy}(y) = 1/2$, and $u^{xy}(z) = 0$ for every $z \in A \setminus \{x, y\}$. We claim that we can assume that $f(u^{xy}) \in \{x, y\}$, since otherwise the distortion is infinite. Indeed, if $f(u^{xy}) = z \notin \{x, y\}$, then consider a utility profiles $\mathbf{u}$ where $u_i \equiv u^{xy}$ for all $i \in N$. Then $\text{win}(f, \mathbf{u}) = \{z\}$, but $\text{sw}(z, \mathbf{u}) = 0$, whereas, say, $\text{sw}(x, \mathbf{u}) = n/2 > 0$.

Let T be a tournament on A, that is, a complete asymmetric binary relation (see, e.g., [8]). For every two alternatives $x, y \in A$, we have that xTy (read: x dominates y) if $f(u^{xy}) = x$, and yTx if $f(u^{xy}) = y$. By our claim above, T is well-defined.

Since the number of pairs of alternatives is $\binom{m}{2} = \frac{m(m-1)}{2}$, by the Pigeonhole Principle there must be an alternative that is dominated by at least $\frac{m-1}{2}$ other alternatives; without loss of generality this alternative is $a \in A$. Let A' be a subset of alternatives of size $\lceil \frac{m-1}{2} \rceil$ such that for all $x \in A'$, xTa. Further, let $A'' = A \setminus (A' \cup \{a\})$ and notice that $|A''| = \lfloor \frac{m-1}{2} \rfloor$. Define a utility function $u^* \in \mathcal{U}$ as follows: for every $x \in A''$, $u^*(x) = 1/|A''|$; for every $x \in A \setminus A''$, $u^*(x) = 0$. Then without loss of generality $f(u^*) = b$, with $b \in A''$, otherwise the distortion is infinite by the same reasoning as above.

Now, we have $|A'| + 1$ blocks of agents, each of size either $\lceil n/(|A'| + 1) \rceil$ or $\lfloor n/(|A'| + 1) \rfloor$. All the agents in the first block, which is at least as large as any other, have the utility function u^* (therefore they vote for b). For each $x \in A'$, there is a block of agents with the utility function u^{ax} (hence they vote for x). Given this utility profile $\mathbf{u}$, b must be among the winners, that is, $b \in \text{win}(f, \mathbf{u})$. We have that

$$\text{sw}(b, \mathbf{u}) \leq \left\lceil \frac{n}{\lceil \frac{m-1}{2} \rceil + 1} \right\rceil \cdot \frac{1}{\lfloor \frac{m-1}{2} \rfloor} \leq \frac{8n}{m^2} \quad ,$$

whereas

$$\text{sw}(a, \mathbf{u}) \geq \left(n - \left\lceil \frac{n}{\lceil \frac{m-1}{2} \rceil + 1} \right\rceil \right) \cdot \frac{1}{2} \geq \frac{n}{6} \quad .$$

The distortion is at least as large as the ratio between the maximum social welfare and the social welfare of a winner with respect to the specific utility profile $\mathbf{u}$, that is,

$$\text{dist}(f) \geq \text{dist}(f, \mathbf{u}) \geq \frac{\text{sw}(a, \mathbf{u}^*)}{\text{sw}(b, \mathbf{u}^*)} = \Omega(m^2) \quad .$$

□

2.2 Randomized Embeddings

Theorem 2 implies that the distortion of any deterministic embedding into Plurality is quite high. Can we do better using randomized embeddings? In general, the answer is definitely positive. However, we start our investigation of randomized embeddings into Plurality with a negative result that holds when the number of agents is small.

Theorem 3. *Let $|N| = n \leq m = |A|$. Then any randomized embedding $f : \mathcal{U} \to \Delta(A)$ into Plurality has distortion $\Omega(m/n)$.*

Proof. Let f be an embedding into Plurality. Consider a utility function $u^* \in \mathcal{U}$ where $u^*(x) = 1/m$ for all $x \in A$. There must exist $x^* \in A$ such that $f(u^*) = x^*$ with probability at most $1/m$.

For $i = 1, 2, ..., n-1$, let $u_i \equiv u^*$ be the utility function of agent i, and let the utility function of agent n be defined by $u_n(x^*) = 1$, $u_n(x) = 0$ for all $x \in A \setminus \{x^*\}$. We have that $\mathrm{sw}(x^*, \mathbf{u}) = 1 + \frac{n-1}{m}$, $\mathrm{sw}(x, \mathbf{u}) = \frac{n-1}{m}$ for any $x \in A \setminus \{x^*\}$.

Now, the probability that $\{x^*\} = \mathrm{win}(f, \mathbf{u})$ is at most the probability that x^* receives a vote from one of the agents $1, 2, \ldots, n-1$, i.e., at most $(n-1)/m$. We conclude that the distortion of f at $\mathbf{u}$ is at least

$$\mathrm{dist}(f, \mathbf{u}) = \frac{1 + \frac{n-1}{m}}{\frac{n-1}{m} \cdot \left(1 + \frac{n-1}{m}\right) + \left(1 - \frac{n-1}{m}\right) \cdot \frac{n-1}{m}} = \frac{m}{2(n-1)} + \frac{1}{2} \quad ,$$

hence $\mathrm{dist}(f) = \Omega(m/n)$. □

We now turn to our presentation of low-distortion embeddings. It turns out that when the number of agents is at least as large as the number of alternatives, a huge reduction in the distortion can be achieved using randomized embeddings. If the number of agents is significantly larger, the distortion can be very close to one. Indeed, consider the following embedding.

Embedding 1 (Naïve randomized embedding into Plurality)**.** Given a utility function $u \in \mathcal{U}$, select alternative $x \in A$ with probability $u(x)$.

The following powerful theorem is our main result.

Theorem 4. *Let $|N| = n \geq m = |A|$, and denote Embedding 1 by f. Then:*

1. *dist*$(f) = \mathcal{O}\left(m^{2m/n}\right)$.
2. *dist*$(f) = \mathcal{O}\left(\sqrt{m}\right)$.
3. *Let $n \geq 3$ and $\epsilon(n, m) = 4\sqrt{\frac{m \ln n}{n}}$. If $\epsilon(n, m) < 1$, then dist*$(f) \leq \frac{1}{1-\epsilon(n,m)}$.

All three bounds on the distortion are required, since each has values of n and m where its guarantees are stronger than the others. Asymptotically, the most powerful bound is the one given in Part 1: it guarantees that the distortion of Embedding 1 is already constant when $n = \Omega(m \ln m)$, that is, when the number of agents is slightly larger than the number of alternatives. This is very close to the necessary condition $n = \Omega(m)$ for obtaining constant distortion that is implied by Theorem 3. Part 1 yields a very weak result for the case $n = m$; in this case, we get by Part 2 that the distortion is $\mathcal{O}\left(\sqrt{m}\right)$. Finally, for large values of n we do not find it sufficient to show that the distortion is *constant*, we want to establish that it is almost one. This does not follow from Part 1 due to the constant hidden in the $\mathcal{O}$ notation. However, from Part 3 we get that for, e.g., $n \geq m^2$, the distortion is $1 + o(1)$.

We now prove the theorem. We will require several results regarding the sums of random variables.

Lemma 5. *Let $X_1, ..., X_n$ be independent heterogeneous Bernoulli trials. Denote by μ the expectation of the random variable $X = \sum_i X_i$. Then (see, e.g., [9] for the different variations on the Chernoff bounds):*

1. *(Jogdeo and Samuels [7])* $\Pr[X < \lfloor \mu \rfloor] < 1/2$.

2. (Lower tail Chernoff bound) For any $\delta \in [0,1]$,

$$\Pr\left[X \leq (1-\delta)\mu\right] \leq \exp\left(-\mu\delta^2/2\right) \quad . \tag{1}$$

3. (Upper tail Chernoff bound) For any $\delta \geq 0$,

$$\Pr\left[X \geq (1+\delta)\mu\right] \leq \left(\frac{e}{1+\delta}\right)^{(1+\delta)\mu} \quad . \tag{2}$$

4. For $\delta \geq 2e-1$,

$$\Pr\left[X \geq (1+\delta)\mu\right] \leq 2^{-(1+\delta)\mu} \quad . \tag{3}$$

5. For $\delta < 2e-1$ *we can use the simplified inequality*

$$\Pr\left[X \geq (1+\delta)\mu\right] \leq \exp\left(-\mu\delta^2/5\right) \quad . \tag{4}$$

Proof of Theorem 4. We prove the theorem's three parts separately. Part 1 is the most straightforward, while Part 2 is similar but slightly more involved, and the proof of Part 3 is quite different and significantly more complicated.

Proof of Part 1. Let $\mathbf{u} \in \mathcal{U}^n$ be a utility profile. First notice that the expected Plurality score of $x \in A$ under the embedding f is $\mathrm{sw}(x, \mathbf{u})$. Let $x^* \in \mathrm{argmax}_{x\in A}\mathrm{sw}(x,\mathbf{u})$ be an alternative with maximum social welfare. We have that $\sum_{x\in A} \mathrm{sw}(x,\mathbf{u}) = n$; by the assumption that $n \geq m$, it follows that $\mathrm{sw}(x^*,\mathbf{u}) \geq n/m \geq 1$. By Part 1 of Lemma 5, with probability at least 1/2 it holds that $\mathrm{sc}(x^*, f, \mathbf{u}) \geq \lfloor \mathrm{sw}(x^*,\mathbf{u})\rfloor$.

Consider some alternative $x \in A$ such that

$$\mathrm{sw}(x,\mathbf{u}) < \frac{\mathrm{sw}(x^*,\mathbf{u})}{2e(4m)^{2m/n}} \quad . \tag{5}$$

We apply the upper tail Chernoff bound (2) to the random variable $\mathrm{sc}(x, f, \mathbf{u})$ with expectation $\mu = \mathrm{sw}(x,\mathbf{u})$ using $(1+\delta)\mu = \lfloor \mathrm{sw}(x^*,\mathbf{u})\rfloor$. By (5) and since $\lfloor \mathrm{sw}(x^*,\mathbf{u})\rfloor > \mathrm{sw}(x^*,\mathbf{u})/2$, we also have $1+\delta > e(4m)^{2m/n}$. Therefore,

$$\Pr[\mathrm{sc}(x,f,\mathbf{u}) \geq \lfloor sw(x^*,\mathbf{u})\rfloor] \leq \left(\frac{1}{(4m)^{2m/n}}\right)^{\lfloor \mathrm{sw}(x^*,\mathbf{u})\rfloor} \leq \left(\frac{1}{(4m)^{2m/n}}\right)^{\mathrm{sw}(x^*,\mathbf{u})/2} \leq \frac{1}{4m} \quad ,$$

where the last inequality follows since $\mathrm{sw}(x^*,\mathbf{u}) \geq n/m$.

By the union bound, the probability that either $\mathrm{sc}(x^*, f, \mathbf{u}) < \lfloor \mathrm{sw}(x^*,\mathbf{u})\rfloor$, or some alternative x that satisfies (5) has $\mathrm{sc}(x, f, \mathbf{u}) \geq \lfloor \mathrm{sw}(x^*,\mathbf{u})\rfloor$, is at most 3/4. Therefore, with probability 1/4 all the winners have social welfare at least $\mathrm{sw}(x^*,\mathbf{u})/(2e(4m)^{2m/n})$. Hence

$$\mathrm{dist}(f,\mathbf{u}) \leq \frac{\mathrm{sw}(x^*,\mathbf{u})}{\frac{1}{4}\cdot\frac{\mathrm{sw}(x^*,\mathbf{u})}{2e(4m)^{2m/n}}} = 8e\cdot(4m)^{2m/n} \quad .$$

Since $n \geq m$, we have that $4^{2m/n} \leq 16$. It follows that the distortion of f is as announced.

Proof of Part 2. The proof is similar to Part 1. Given $\mathbf{u} \in \mathcal{U}^n$, we once again denote by x^* the alternative with maximum social welfare, and we let $L \subset A$ be the set of alternatives with social welfare smaller than $\mathrm{sw}(x^*, \mathbf{u})/(3e\sqrt{m})$, that is,

$$L = \left\{ x \in A : \ \mathrm{sw}(x, \mathbf{u}) < \frac{\mathrm{sw}(x^*, \mathbf{u})}{3e\sqrt{m}} \right\} \quad .$$

If $L = \emptyset$ then the claim follows trivially, hence we can restrict our attention to three cases. In all three cases we demonstrate that with constant probability no alternative in L is among the winners, that is, with probability bounded away from zero an alternative with social welfare at least $\mathrm{sw}(x^*, \mathbf{u})/(3e\sqrt{m})$ is elected, which directly yields the bound on the distortion.

Case 1: $\mathrm{sw}(x^*, \mathbf{u}) < 2$ and $|L| = 1$. Since $n \geq m$ it also holds that $\mathrm{sw}(x^*, \mathbf{u}) \geq 1$ and, hence, by Jogdeo and Samuels the probability that $\mathrm{sc}(x^*, f, \mathbf{u}) = 0$ is at most $1/2$. Let x be the element of L. Since $\mathrm{sw}(x^*, \mathbf{u}) < 2$, it also holds that $\mathrm{sw}(x, \mathbf{u}) < 2/(3e\sqrt{m})$ and, by Markov's inequality, we have that the probability that $\mathrm{sc}(x, f, \mathbf{u}) \geq 1$ is at most $2/(3e\sqrt{m}) < 1/4$.

Case 2: $\mathrm{sw}(x^*, \mathbf{u}) < 2$ and $|L| > 1$. For each $i \in N$, let X_i be a random variable such that $X_i = 1$ if $f(u_i) \notin L$, that is, agent i votes for an alternative not in L. We have that

$$\sum_{i \in N} X_i = \sum_{x \notin L} \mathrm{sc}(x, f, \mathbf{u}) \quad .$$

The sum $\sum_{x \notin L} \mathrm{sc}(x, f, \mathbf{u})$ has expectation at least

$$m - \frac{2}{3e\sqrt{m}} \cdot |L| > m - |L| + 1 \quad .$$

By Jogdeo and Samuels (using the fact that the X_i are independent) with probability at least $1/2$ it holds that $\sum_{x \notin L} \mathrm{sc}(x, f, \mathbf{u}) \geq m - |L| + 1$. If so then by the Pigeonhole Principle there exists some alternative $x^0 \in A \setminus \{L\}$ which has $\mathrm{sc}(x^0, f, \mathbf{u}) \geq 2$.

Now, consider some alternative $x \in L$. We apply Equation (2) to the random variable $\mathrm{sc}(x, f, \mathbf{u})$ with expectation $\mu = \mathrm{sw}(x, \mathbf{u})$ using $(1+\delta)\mu = 2$. Since $\mu \leq 2/(3e\sqrt{m})$, it holds that $1+\delta > 3e\sqrt{m}$. We conclude that

$$\Pr\left[\mathrm{sc}(x, f, \mathbf{u}) \geq 2\right] \leq \left(\frac{1}{3\sqrt{m}}\right)^2 = \frac{1}{9m} \quad .$$

By the union bound the probability that some alternative in L is among the winners is at most $(1/9m) \cdot m + 1/2 = 11/18$.

Case 3: $\mathrm{sw}(x^*, \mathbf{u}) \geq 2$. By Jogdeo and Samuels, the probability that $\mathrm{sc}(x^*, f, \mathbf{u}) < \lfloor \mathrm{sw}(x^*, \mathbf{u}) \rfloor$ is at most $1/2$. Next we consider some alternative $x \in L$. We apply (2) to the random variable $\mathrm{sc}(x, f, \mathbf{u})$ with expectation $\mu = \mathrm{sw}(x, \mathbf{u})$ using

$$(1+\delta)\mu = \lfloor \mathrm{sw}(x^*, \mathbf{u}) \rfloor \quad .$$

Since

$$\mu < \frac{\mathrm{sw}(x^*, \mathbf{u})}{3e\sqrt{m}} \leq \frac{\lfloor \mathrm{sw}(x^*, \mathbf{u}) \rfloor}{2e\sqrt{m}} \quad ,$$

it holds that $1+\delta > 2e\sqrt{m}$. We conclude that

$$\Pr\left[\mathrm{sc}(x,f,\mathbf{u}) \geq 2\right] \leq \left(\frac{1}{2\sqrt{m}}\right)^{\lfloor \mathrm{sw}(x*,\mathbf{u}) \rfloor} \leq \left(\frac{1}{2\sqrt{m}}\right)^{2} = \frac{1}{4m} \quad .$$

Similarly to Case 2, we apply the union bound and conclude that the probability that $\mathrm{sc}(x,f,\mathbf{u}) \geq \mathrm{sc}(x^*,f,\mathbf{u})$ for some $x \in L$ is at most $(1/4m)\cdot m + 1/2 = 3/4$.

Proof of Part 3. Given $\mathbf{u} \in \mathcal{U}^n$ we consider the alternative $x^* \in A$ with the maximum social welfare. Denote by $L \subset A$ the set of alternatives with social welfare at most $\mathrm{sw}(x^*,\mathbf{u}) - \left(\sqrt{2}+\sqrt{5}\right)\sqrt{\mathrm{sw}(x^*,\mathbf{u})\ln n}$. We will show that the probability that there exists $x \in L$ such that $\mathrm{sc}(x,f,\mathbf{u}) \geq \mathrm{sc}(x^*,f,\mathbf{u})$ is at most m/n. Specifically, we will establish that the probability that either

$$\mathrm{sc}(x^*,f,\mathbf{u}) \leq \mathrm{sw}(x^*,\mathbf{u}) - \sqrt{2\mathrm{sw}(x^*,\mathbf{u})\ln n}$$

or

$$\mathrm{sc}(x,f,\mathbf{u}) \geq \mathrm{sw}(x^*,\mathbf{u}) - \sqrt{2\mathrm{sw}(x^*,\mathbf{u})\ln n}$$

for some $x \in L$ is at most m/n.

We first apply bound (1) to the random variable $\mathrm{sc}(x^*,f,\mathbf{u})$ with $\delta = \sqrt{(2\ln n)/(\mathrm{sw}(x^*,\mathbf{u}))}$. Since the expectation of $\mathrm{sc}(x^*,f,\mathbf{u})$ is $\mathrm{sw}(x^*,\mathbf{u})$ we have that

$$\Pr\left[\mathrm{sc}(x^*,f,\mathbf{u}) \leq \mathrm{sw}(x^*,\mathbf{u}) - \sqrt{2\mathrm{sw}(x^*,\mathbf{u})\ln n}\right] \leq \frac{1}{n} \quad .$$

Next we consider an alternative $x \in L$ and the random variable $\mathrm{sc}(x,f,\mathbf{u})$. This variable has expectation $\mu < \mathrm{sw}(x^*,\mathbf{u}) - \left(\sqrt{2}+\sqrt{5}\right)\sqrt{\mathrm{sw}(x^*,\mathbf{u})\ln n}$. We apply the upper tail Chernoff bound with δ such that

$$(1+\delta)\mu = \mathrm{sw}(x^*,\mathbf{u}) - \sqrt{2\mathrm{sw}(x^*,\mathbf{u})\ln n} \quad .$$

Clearly, $\delta\mu > \sqrt{5\mathrm{sw}(x^*,\mathbf{u})\ln n}$. If $\delta \geq 2e-1$, Equation (3) yields

$$\Pr\left[\mathrm{sc}(x,f,\mathbf{u}) \geq \mathrm{sw}(x^*,\mathbf{u}) - \sqrt{2\mathrm{sw}(x^*,\mathbf{u})\ln n}\right] \leq 2^{-\left(\mathrm{sw}(x^*,\mathbf{u}) - \sqrt{2\mathrm{sw}(x^*,\mathbf{u})\ln n}\right)} \leq 2^{-(16-4\sqrt{2})\ln n} \leq \frac{1}{n}$$

where the second inequality follows from the fact that $\mathrm{sw}(x^*,\mathbf{u}) \geq n/m$ and since the assumption $\epsilon(n,m) < 1$ implies that $n/m > 16\ln n$.

If $\delta < 2e-1$, Equation (4) yields

$$\Pr\left[\mathrm{sc}(x,f,\mathbf{u}) \geq \mathrm{sw}(x^*,\mathbf{u}) - \sqrt{2\mathrm{sw}(x^*,\mathbf{u})\ln n}\right] \leq \exp\left(-\frac{(\delta\mu)^2}{5\mu}\right) < \exp\left(-\frac{\mathrm{sw}(x^*,\mathbf{u})\ln n}{\mu}\right) \leq \frac{1}{n} \quad .$$

By the union bound, we have that the probability that some of the undesired events happen is at most m/n (there are at most m such events). Hence, with probability at least $1-m/n$ some alternative $x \in A \setminus L$ is the winner, and the expected score of the worst winner is at least

$$\begin{aligned}
&\left(\text{sw}(x^*, \mathbf{u}) - \left(\sqrt{2}+\sqrt{5}\right)\sqrt{\text{sw}(x^*, \mathbf{u}) \ln n}\right)\left(1 - \frac{m}{n}\right) \\
&= \text{sw}(x^*, \mathbf{u})\left(1 - \left(\sqrt{2}+\sqrt{5}\right)\sqrt{\frac{\ln n}{\text{sw}(x^*, \mathbf{u})}}\right)\left(1 - \frac{m}{n}\right) \\
&\geq \text{sw}(x^*, \mathbf{u})\left(1 - \left(\sqrt{2}+\sqrt{5}\right)\sqrt{\frac{m \ln n}{n}}\right)\left(1 - \sqrt{\frac{m \ln n}{16n}}\right) \\
&\geq \text{sw}(x^*, \mathbf{u})\left(1 - \left(\sqrt{2}+\sqrt{5}+\frac{1}{4}\right)\sqrt{\frac{m \ln n}{n}}\right) \\
&\geq \text{sw}(x^*, \mathbf{u})\left(1 - 4\sqrt{\frac{m \ln n}{n}}\right) \\
&= \text{sw}(x^*, \mathbf{u})\left(1 - \epsilon(n,m)\right) \quad .
\end{aligned}$$

The second transition holds since $n \geq 3$ together with $\epsilon(n,m) < 1$ imply that

$$\frac{m}{n} \leq \sqrt{\frac{m}{16n \ln n}} \leq \sqrt{\frac{m \ln n}{16n}}$$

and, furthermore, $\text{sw}(x^*, \mathbf{u}) \geq n/m$. The third transition follows from the inequality $(1-\alpha)(1-\beta) \geq (1-\alpha-\beta)$ for any $\alpha, \beta \in [0,1]$. ☐

Our final result regarding embeddings into Plurality asserts that the upper bound of $\mathcal{O}\left(\sqrt{m}\right)$ for the case of $n = m$, which follows from Part 2 of Theorem 4, is almost tight. This case is especially interesting since for slightly larger values of n the distortion is constant.

Theorem 6. *Let $|N| = n = m = |A|$, and denote Embedding 1 by f. Then dist$(f) = \Omega\left(\sqrt{\frac{m}{\ln m}}\right)$.*

Proof. The main idea is the construction of a utility profile with "heavy" (high social welfare) alternatives and "light" (low social welfare) alternatives; the heavy alternatives have social welfare and score of exactly two, whereas the light alternatives have low social welfare and expected score. However, since there are many light alternatives, with high probability at least one such alternative has a score of two.

Formally, let t, λ, and k be integers to be defined later. Consider an instance with $N = N' \cup N''$, where $|N'| = t$ and $|N''| = \lambda$ (i.e., $n = t + \lambda$). Furthermore, let $A = A' \cup A''$, where $|A'| = t/2$ and $|A''| = k\lambda/2$ (i.e., $m = t/2 + k\lambda/2$). It also holds that $m = n$, which implies that $m = \lambda(k-1)$. We construct a utility profile $\mathbf{u} \in \mathcal{U}^n$ as follows. Each $x \in A'$ has utility equal to 1 with respect to the utility functions of exactly two agents in N', that is, for all $x \in A'$, $\text{sw}(x, \mathbf{u}) = 2$. Each $x \in A''$ has utility $1/k$ with respect to the utility functions of exactly two of the agents in N'', hence for all $x \in A''$, $\text{sw}(x, \mathbf{u}) = 2/k$.

The probability that an alternative $x \in A''$ satisfies $\text{sc}(x, f, \mathbf{u}) < 2$ is $1 - \frac{1}{k^2}$. Moreover, the probability that $\text{sc}(x, f, \mathbf{u}) < 2$ given that a subset other alternatives in A'' have score less than two is at most $1 - \frac{1}{k^2}$. Therefore (by an implicit application of the chain rule) the probability that no $x \in A''$ has score two is at most

$$\left(1 - \frac{1}{k^2}\right)^{k\lambda/2} \leq \exp\left(-\frac{\lambda}{2k}\right) \quad .$$

Selecting $\lambda = 2\lceil k \ln k \rceil$, we have that the probability that no alternative in A'' has score two is at most $1/k$. It follows that the distortion is at least

$$\frac{2}{2 \cdot \frac{1}{k} + \frac{2}{k} \cdot \left(1 - \frac{1}{k}\right)} \geq \frac{k}{2} \quad .$$

Clearly $m = \mathcal{O}(k^2 \ln k)$ and, hence, the bound on the distortion follows. □

3 Embeddings into Approval

Under the Approval rule, each agent approves a subset of the alternatives. Each approved alternative receives one point. The set of winners includes the alternatives with most points, summed over all the agents.

We must reformulate some of our definitions in order to apply our notions to Approval voting. A deterministic embedding into Approval is a function $f : \mathcal{U} \to 2^A$, where 2^A is the powerset of alternatives. In words, an agent with a utility function u approves each of the alternatives in $f(u)$. The (Approval) score of an alternative is redefined to be

$$\mathrm{sc}(x, f, \mathbf{u}) = |\{i \in N \ : \ x \in f(u_i)\}| \quad .$$

A randomized embedding is a function $f : \mathcal{U} \to \Delta(2^A)$. The rest of the definitions (in particular, the definition of distortion) are the same as before.

3.1 Deterministic Embeddings

In Section 2 we have seen that no deterministic embedding into Plurality can achieve distortion better than $\Omega(m^2)$ (Theorem 2). As it turns out, better results can be achieved with respect to Approval. Indeed, consider the following Embedding.

Embedding 2 (Deterministic embedding into Approval)**.** Given a utility function u, approve the subset of alternatives $x \in A$ such that $u(x) \geq 1/m$.

The following straightforward result establishes that the distortion of this embedding is $\mathcal{O}(m)$.

Theorem 7. *Let $|N| = n$, $|A| = m$, and denote Embedding 2 by f. Then dist$(f) \leq 2m - 1$.*

Proof. Let $\mathbf{u}$ be a utility profile. Let $x \in \mathrm{win}(f, \mathbf{u})$ be a winning alternative, and let $x^* \in A$ be an alternative which maximizes the social welfare. Alternative x^* has $u_i(x^*) < 1/m$ with respect to $n - \mathrm{sc}(x^*, f, \mathbf{u})$ agents i, and has utility at most one with respect to $\mathrm{sc}(x^*, f, \mathbf{u})$ agents. Hence,

$$\begin{aligned}
\mathrm{sw}(x^*, \mathbf{u}) &< \mathrm{sc}(x^*, f, \mathbf{u}) + (n - \mathrm{sc}(x^*, f, \mathbf{u})) \cdot \frac{1}{m} \\
&= \frac{n}{m} + \left(1 - \frac{1}{m}\right) \cdot \mathrm{sc}(x^*, f, \mathbf{u}) \\
&\leq \left(2 - \frac{1}{m}\right) \cdot \mathrm{sc}(x, f, \mathbf{u}) \leq (2m - 1) \cdot \mathrm{sw}(x, \mathbf{u}) \quad .
\end{aligned}$$

The third transition holds since x is a winning alternative (and, hence, $\mathrm{sc}(x^*, f, \mathbf{u}) \leq \mathrm{sc}(x, f, \mathbf{u})$) and also has score at least n/m (since, by the definition of the embedding, at least one alternative is approved by each agent). The last transition follows from the definition of the embedding, which implies that $\mathrm{sc}(x, f, \mathbf{u}) \leq m \cdot \mathrm{sw}(x, \mathbf{u})$. □

Unfortunately, it is impossible to design low-distortion deterministic embeddings into Approval. In fact, the following theorem asserts that the simple Embedding 2 is asymptotically optimal.

Theorem 8. *Let $|N| = n \geq 2$, $|A| = m \geq 3$, and let $f : \mathcal{U} \to 2^A$ be a deterministic embedding into Approval. Then dist$(f) \geq (m-1)/2$.*

Proof. Let f be a deterministic embedding into Approval. Consider the utility function $u^1 \in \mathcal{U}$ where $u^1(a) = 0$, $u^1(x) = 1/(m-1)$ for all $x \in A \setminus \{a\}$. We can assume that $f(u^1)$ does not approve a and approves at least one $x^* \in A \setminus \{a\}$, otherwise we get infinite distortion by considering the utility profile where $u_i \equiv u^1$ for all $i \in N$. Without loss of generality $f(u^1)$ approves $b \in A \setminus \{a\}$.

Now, let $u^2 \in \mathcal{U}$ be defined by $u^2(a) = 1$, $u^2(x) = 0$ for all $x \in A \setminus \{a\}$ (in particular, $u^2(b) = 0$). We define a utility profile $\mathbf{u} \in \mathcal{U}^n$ by setting $u_i \equiv u^1$ for $\lceil n/2 \rceil$ agents i, and $u_i \equiv u^2$ for $\lfloor n/2 \rfloor$ agents i. By the argument above it holds that $b \in \text{win}(f, \mathbf{u})$, but (using the assumption on the size of n and m) it holds that

$$\text{dist}(f, \mathbf{u}) \geq \frac{\text{sw}(b, \mathbf{u})}{\text{sw}(a, \mathbf{u})} \geq \frac{m-1}{2} \quad .$$

We conclude that $\text{dist}(f) \geq (m-1)/2$. □

3.2 Randomized Embeddings

In the context of deterministic embeddings, we have seen that there is a gap between the distortion of embeddings into Approval and embeddings into Plurality. It turns out that there is also a huge gap with respect to randomized embeddings, when the number of agents is very small.

Indeed, consider the randomized embedding f into Approval that, with probability 1/2, approves an alternative with maximum utility, and with probability 1/2 approves all the alternatives. Further, assume that $N = \{1, 2\}$, and let $\mathbf{u} \in \mathcal{U}^2$. Without loss of generality there exists $x^* \in A$ such that $u_1(x^*) \geq u_i(x)$ for all $x \in A$ and all $i \in N$. Then clearly, for every $x \in A$, $\text{sw}(x, \mathbf{u}) \leq 2 \cdot \text{sw}(x^*, \mathbf{u})$. Moreover, it holds that $\text{win}(f, \mathbf{u}) = \{x^*\}$ with probability at least 1/4. Hence, the distortion of this embedding is at most eight, i.e., constant. This reasoning can easily be extended to obtain constant distortion with respect to any constant n. Compare this result with Theorem 3.

However, as before, we are mostly interested in the case of a large number of agents. Crucially, every embedding into Plurality can also be seen as an embedding into Approval, where for every utility function exactly one alternative is approved. Hence, the powerful positive result regarding Embedding 1, namely Theorem 4, also holds with respect to Approval.

It is natural to consider the following embedding into Approval.

Embedding 3 (Naïve randomized embedding into Approval)**.** Given a utility profille $\mathbf{u} \in \mathcal{U}^n$, independently approve each alternative $x \in A$ with probability $u(x)$.

So, in contrast to Embedding 1 into Plurality, under Embedding 3 multiple alternatives can be approved. However, the expected score of an alternative under both embeddings is identical. This implies (not directly) that Theorem 4, and even the lower bound given in Theorem 6, apply to Embedding 3 as well.

It remains open whether there is a gap in the distortion of randomized embeddings into Plurality and randomized embeddings into Approval when $n \geq m$. Interestingly enough, the lower bound of $\Omega(\sqrt{m/\ln m})$ for $n = m$ (Theorem 6) also holds with respect to some natural embeddings into

Approval which may approve multiple alternatives, such as Embedding 4 that is considered in the next section.

4 Experimental results

In this section, we present experimental results concerning representative randomized embeddings into Plurality and Approval. Our aim is to shed some light on the enigmatic case in which the number of agents is smaller than the number of alternatives. The main message from our experiments is that efficiency with respect to the distortion can be obtained by randomized embeddings in this case as well. Recall that our positive result (Theorem 4) does not apply to this case.

We remark that, since the definition of distortion involves all utility profiles with specific numbers of agents and alternatives as well as the expectation of the social welfare of the winning alternative, we should not expect its exact measurement in our experiments. Instead, we will approximate the distortion of embeddings by considering many utility profiles that are produced according to carefully selected probability distributions, which can serve as strong adversaries for our embeddings, and by considering the execution of the embeddings many times on each utility profile. Our experimental setting also allows us to make interesting observations about the efficiency of embeddings on particular probability distributions of utility profiles; in this context, we will examine distortion measurements that deviate from the standard definition of distortion (see below) and are specific to particular probability distributions.

In more detail, we consider utility profiles that are produced randomly according to a family of different probability distributions. The probability distributions are defined as follows for values of parameter τ in $[1, +\infty)$.

τ-biased probability distributions. A specific alternative x is identified. For each agent, we pick a random value in the range $[0, 1]$ for each alternative. We multiply the value corresponding to the specific alternative x by τ. These values are then normalized (so that their sum is unity) in order to compute the utility function of the agent over the alternatives.

We use the term τ-biased utility profiles to refer to utility profiles that are produced randomly according to the τ-biased probability distribution. Such utility profiles have the following properties:

- The expected social welfare of all alternatives except x is the same.
- The expected social welfare of alternative x is τ times the expected social welfare of any other alternative.

The rationale behind the selection of these probability distributions is that they can challenge our embeddings (by producing utility profiles that are difficult to handle efficiently) and reveal empirical statements of some generality about their distortion. Intuitively, the social welfare of an alternative will be concentrated around its expectation. In the extreme case of τ-biased profiles for high values of τ, most agents will also have a high utility for a particular alternative (i.e., alternative x). This alternative probably has the highest social welfare and should be the winner in any low-distortion embedding. Indeed, our experimental results will verify this observation. On the other hand, in τ-biased utility profiles for small values of τ, it may not be apparent that the alternative with the highest social welfare is the one that would be selected by the voting rule; recall that the number of alternatives is large, the randomized embeddings make random choices and, furthermore, the number of agents is small. The ability of our embeddings to identify alternatives with high social welfare (for different values of τ) will reflect their efficiency in terms of the distortion.

In the experiments presented below we have run Embedding 1 into Plurality on τ-biased utility profiles. We have also considered the following embedding into Approval which tries to exploit the extra flexibility Approval provides by allowing each agent to approve any number of alternatives.

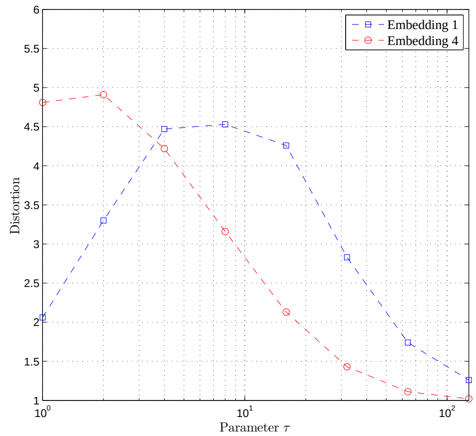
Embedding 4. For any agent, pick a value in [0,1] uniformly at random and approve all alternatives (if any) that have utility higher than this value.

We remark that Embedding 3 could be a natural embedding to consider here but, not surprisingly, the results obtained are similar to those of Embedding 1. Our upper bound analysis (i.e., the statement and proof of Theorem 4) can be easily seen to extend to Embedding 4 as well but it does not capture the case in which the number of agents is smaller than the number alternatives.

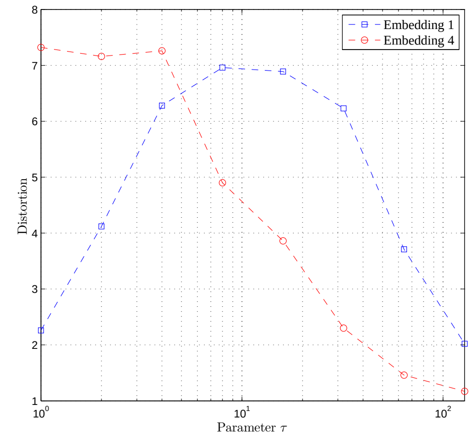
In our experiments, the distortion at a utility profile is computed using Definition 1.2 as the average social welfare of the winner after 1000 executions of the randomized embedding over the maximum social welfare among all alternatives in the particular profile. Ties among different winning alternatives are broken in favor of the alternative with the minimum social welfare as Definition 1.2 suggests. The results from a representative set of experiments are depicted in Figure 1. The four plots contain results from the application of Embeddings 1 and 4 on τ-biased utility profiles (for values of τ that are powers of 2 and lie between 1 and 128) with 64 or 128 alternatives and 8 or 16 agents. Here, we have used a distortion measure that is specific to τ-biased utility profiles for a given value of τ and particular numbers of agents and alternatives. More precisely, for each point in the four plots of Figure 1, the distortion value was computed as the maximum distortion in 1000 different τ-biased utility profiles. This gives us a more refined measure of the efficiency of randomized embeddings as a function of the values of τ.

The results provide information the theoretical analysis cannot provide unless it becomes very detailed and adapted to the particular probability distributions. In all experiments we have performed with Embedding 1 for Plurality, the distortion is non-monotonic with respect to τ. There exists a particular value (or range of values) of τ for which the distortion is maximized (see Figure 1). The low distortion values for high values of τ can be easily explained. According to the definition of τ-biased utility profiles, many agents are expected to have a significantly higher utility for the particular alternative x. Consequently, Embedding 1 will translate such utilities into a vote for alternative x, and hence x is often selected. For small values of τ, the social welfare of the alternatives is around the (same, more or less) expectation, and Embedding 1 computes a winner with social welfare that is not far from the maximum one. τ-biased utility profiles for intermediate values of τ are the most difficult to handle. However, in all cases the distortion is relatively low.

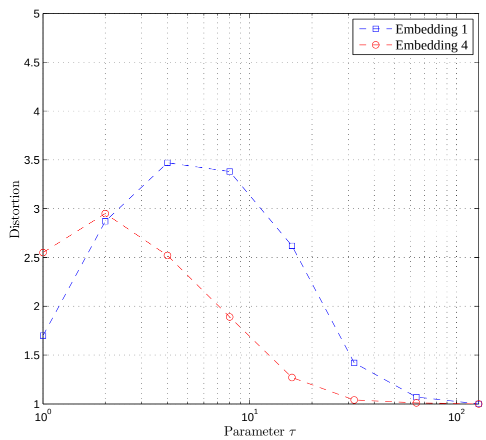
Interestingly, the experimental results indicate that the behavior of Embedding 4 on τ-biased utility profiles is different. Our observation for Embedding 1 for high values of τ applies to Embedding 4 as well. On the other hand, Embedding 4 seems to have its highest distortion for small values of τ (unlike Embedding 1, which efficiently handles such utility profiles). This is due to the fact that, by its definition, Embedding 4 cannot help an agent distinguish between alternatives that have comparable utility since (unlike Embedding 1) it may translate the utility function of an agents to approvals for all such alternatives. This effect almost vanishes as τ increases and, for most values of the parameter τ, Embedding 4 significantly outperforms Embedding 1, quickly reaching optimal distortion. This last phenomenon is more apparent when the number of agents is large (compare Figures 1(c) and 1(d) with Figures 1(a) and 1(b), respectively) and implies that the extra flexibility of Embedding 4 is beneficial in this case.



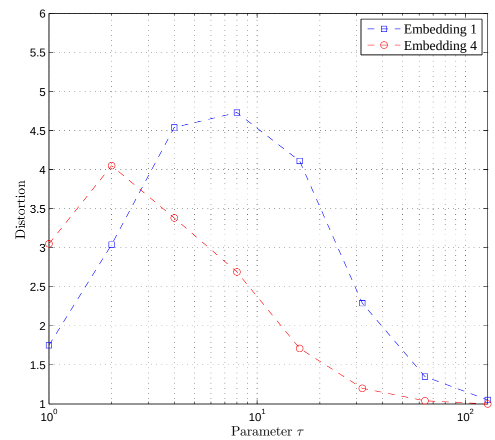
(a) 64 alternatives, 8 agents.



(b) 128 alternatives, 8 agents.



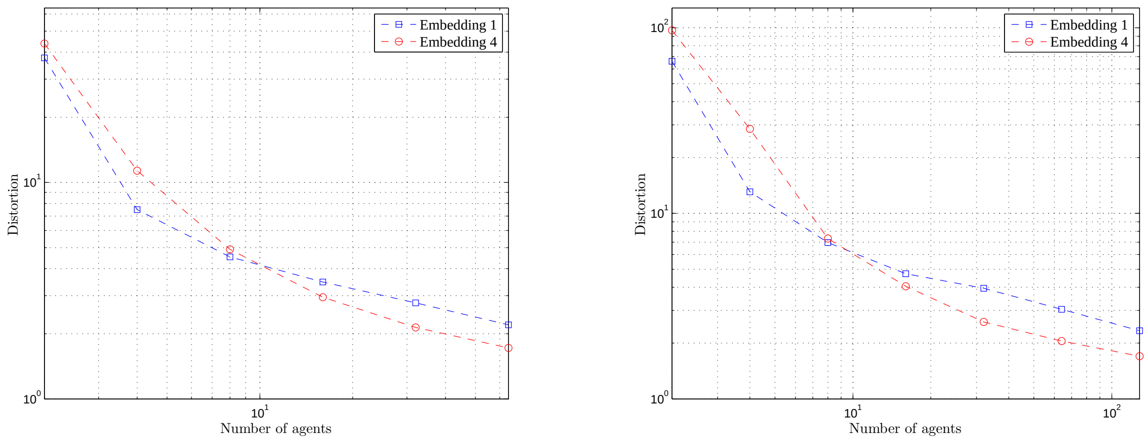
(c) 64 alternatives, 16 agents.



(d) 128 alternatives, 16 agents.

Figure 1: Experiments in profiles with 64 or 128 alternatives and 8 or 16 agents. The distortion is a function of parameter τ which takes as values the powers of 2 from 1 to 128. Note that the scale of the x-axis is logarithmic.

A summary of the results of our experiments with profiles with 64 or 128 alternatives and a number of agents that is a power of 2 between 2 and 64 or 128, respectively, is depicted in the two plots of Figure 2. These results are similar in spirit to our upper bound statements (e.g., Theorem 4) in the sense that the distortion bounds are worst-case among utility profiles produced according to different probability distributions. Each point in these plots represents the maximum distortion observed in τ-biased profiles for all different values of τ that are powers of 2 with the particular number of agents and number of alternatives corresponding to the point. For example, the point of Figure 2(b) that corresponds to the execution of Embedding 1 with 16 agents has a distortion value equal to the maximum (i.e., 4.73) among the eight distortion values for this embedding in Figure 1(d) (that is obtained for $\tau = 8$). So, the distortion value at each point in the plots of Figure 2 is



(a) 64 alternatives.

(b) 128 alternatives.

Figure 2: Summary of experiments in profiles with 64 or 128 alternatives. The distortion is a function of the number of agents (which takes as values the powers of 2 from 2 to the number of alternatives). Note that the scale in both axes is logarithmic.

the maximum distortion observed in 8000 different utility profiles. The remaining points have been produced by running the two embeddings in utility profiles with appropriate parameters.

The results suggest a threshold behavior with respect to the relative performance of the two Embeddings: Embedding 1 outperforms Embedding 4 in profiles with small number of agents, whereas the opposite is true when the number of agents is high. The transition takes place when the number of agents goes from 8 to 16 and pinpoints the threshold at which the extra flexibility that Approval provides (and Embedding 4 exploits) becomes beneficial. Alternatively, this transition in the behavior of Embeddings 1 and 4 can be observed by examining the maximum distortion (over all values of τ) of each embedding in the four plots of Figure 1 (and can be explained by the discussion on the behavior of Embedding 4 above). However, in general, the results indicate that, at least for the particular family of probability distributions of utility profiles, the distortion of both embeddings is $O(m/n)$. This claim is supported by the 13 pairs of points depicted in Figure 2 as well as by other experiments on utility profiles with intermediate numbers of alternatives and agents that are not reported here. This bound matches asymptotically the theoretical lower bound of Theorem 3 and is superior to the theoretical upper bound of Theorem 4 (for $n = m$). The latter is to be expected since the lower bound construction used in the proof of Theorem 3 is unlikely to be produced by the τ-biased probability distributions.

5 Embeddings into Veto

Under the Veto rule, each agent vetoes a single (presumably least preferred) alternative. The set of winners includes all the alternatives that are vetoed the least number of times. Equivalently, each agent awards one point to all the alternatives except one, and the alternatives with most points are the winners.

The Veto rule can be interpreted as a *scoring rule*. Such a rule is defined by a vector of real numbers $(\alpha_1, \alpha_2, ..., \alpha_m)$ with $\alpha_1 = 1 \geq \alpha_2 \geq ... \geq \alpha_{m-1} \geq \alpha_m = 0$. Let $\mathcal{L}(A)$ denote the set of rankings over A. The vote of each agent is an element of $\mathcal{L}(A)$. The number of points awarded by an agent to the alternative ranked in the k'th position is α_k. Veto then is the scoring rule defined by $(1, \ldots, 1, 0)$. Plurality is the scoring rule defined by $(1, 0, \ldots, 0)$.

In this section we are mostly interested in the Veto rule, due to its low communication costs ($\log m$ bits per agent). However, one can think of other scoring rules with low communication costs, e.g., 2-approval defined by $(1, 1, 0, \ldots, 0)$ or 2-anti-approval defined by $(1, \ldots, 1, 0, 0)$. Hence, we will formulate some of our results for scoring rules in general. Note that a deterministic embedding into a scoring rule R is a function $f : \mathcal{U} \to \mathcal{L}(A)$, whereas a randomized embedding is a function $f : \mathcal{U} \to \Delta(\mathcal{L}(A))$.

5.1 Deterministic Embeddings

The Plurality and Veto rules are closely related in the sense that agents must award an equal number of points to almost all alternatives, and therefore cannot make a distinction in their votes between very desirable and very undesirable alternatives. However, this turns out to be a more acute problem under Veto, since agents cannot even single out one good alternative. The following definition allows us to quantify this property.

Definition 9. Let R be a scoring rule with score vector $(\alpha_1, \alpha_2, ..., \alpha_m)$ over m alternatives with $1 = \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_{m-1} \geq \alpha_m = 0$. The *decisiveness* d_R of R is defined as $d_R = m - \sum_{i=1}^{m} \alpha_i$.

Observe that the decisiveness of a scoring rule lies between 1 (for Veto) to $m-1$ (for Plurality). Procaccia and Rosenschein [11] (implicitly) relate the distortion of the naïve deterministic embedding to decisiveness. The naïve deterministic embedding simply computes a non-increasing ordering of the alternatives with respect to their utilities, breaking ties among the alternatives according to a predefined rule (e.g., lexicographically).

Theorem 10 (Procaccia and Rosenschein [11])**.** *The naive deterministic embedding into a scoring rule R with decisiveness at most $m-2$ has infinite distortion.*

We can extend the above impossibility result to any deterministic embedding.

Theorem 11. *Let $|N| = n \geq 1$ and $|A| = m \geq 3$, and let $f : \mathcal{U} \to A$ be a deterministic embedding into a scoring rule with $\alpha_1 = \alpha_2$. Then dist$(f) = \infty$.*

Proof. Let $a \in A$ and $\mathbf{u}$ be a utility profile such that for all $i \in N$, $u_i(a) = 1$, and $u_i(x) = 0$ for all $x \in A \setminus \{a\}$. Let f be an embedding into a scoring rule with $\alpha_1 = \alpha_2 = 1$. Let $y \in A \setminus \{a\}$ that is ranked in one of the first two positions of the ranking $f(u)$. Then, y has a score of n and, hence, $y \in \text{win}(f, \mathbf{u})$. Since $\text{sw}(a, \mathbf{u}) = n$ and $\text{sw}(y, \mathbf{u}) = 0$, it follows that the distortion of f is infinite. □

It follows that for all the scoring protocols that could be of interest due to their low communication requirements, deterministic embeddings have infinite distortion.

5.2 Randomized Embeddings

First, observe that if $n < m-1$, then any randomized embedding into Veto has infinite distortion, for reasons similar to the deterministic case (Theorem 11). Indeed, consider n agents with utility 1 for alternative $x^* \in A$ and 0 for the remaining $m-1$ alternatives. At least one of these $m-1$ alternatives is not vetoed, and hence it is included in the set of winners. However, for larger values of n it is possible to obtain positive results; we consider the following embedding into any scoring rule.

Embedding 5 (Randomized embedding into scoring rules)**.** Given a utility function $u \in \mathcal{U}$, select an alternative $x \in A$ with probability $u(x)$ to be ranked first; denote the selected alternative by x^*. Now, complete the ranking at positions $2, \ldots, m$ by selecting a random permutation among the alternatives in $A \setminus \{x^*\}$.

For Veto Embedding 5 reduces to the following scheme. As in the general embedding, select an alternative $x \in A$ with probability $u(x)$, and denote the selected alternative by x^*. Now, the vetoed alternative $f(u)$ is selected uniformly at random from $A \setminus \{x^*\}$ (that is, each alternative in $A \setminus \{x^*\}$ is selected with probability $1/(m-1)$). Interestingly, for Plurality Embedding 5 reduces to Embedding 1.

We have the following upper bound on the distortion of Embedding 5.

Theorem 12. *Let $|N| = n \geq m = |A|$, $n \geq 3$, and denote by f the Embedding 5 to a scoring rule R with decisiveness $d_R \in [1, m-1]$. Furthermore, let*

$$\epsilon(n,m) = \frac{2m(m-1)}{d_R}\sqrt{\frac{\ln n}{n}} \quad .$$

If $\epsilon(n,m) < 1$, then dist$(f) \leq \frac{1}{1-\epsilon(n,m)}$.

The proof of this theorem is along similar lines to the proof of Part 3 of Theorem 4. The main difference is that, instead of using Lemma 5 which applies to sums of Bernoulli trials, we now have to use a more general inequality due to Hoeffding [6] which applies to sums of independent heterogeneous random variables (taking values in the range $[0,1]$). We note that, since this inequality is more general, it yields a weaker bound for Plurality than the one obtained in Part 3 of Theorem 4.

Lemma 13 (Hoeffding [6])**.** *Let $X_1, ..., X_n$ be independent heterogeneous random variables with $X_i \in [0,1]$. Denote by μ the expectation of the random variable $X = \sum_i X_i$. Then for any $\lambda > 0$,*

$$\Pr[|X-\mu| \geq \lambda] \leq \exp\left(-\frac{2\lambda^2}{n}\right) \quad .$$

Proof of Theorem 12. Let $\lambda = \frac{1}{2}\sqrt{n \ln n}$. Given $\mathbf{u} \in \mathcal{U}^n$, consider the alternative x^* with maximum social welfare and denote by L the set of alternatives with social welfare less than $\mathrm{sw}(x^*, \mathbf{u}) - 2\lambda\frac{m-1}{d_R}$. We will show that the probability that either

$$\mathrm{sc}(x^*, f, \mathbf{u}) \leq \mathbb{E}[\mathrm{sc}(x^*, f, \mathbf{u})] - \lambda$$

or

$$\mathrm{sc}(x, f, \mathbf{u}) \geq \mathbb{E}[\mathrm{sc}(x^*, f, \mathbf{u})] - \lambda$$

for some $x \in L$ is at most $m/\sqrt{n}$.

Using the Hoeffding bound for the random variable $\mathrm{sc}(x^*, f, \mathbf{u})$, we have

$$\Pr\left[\mathrm{sc}(x^*, f, \mathbf{u}) \le \mathbb{E}[\mathrm{sc}(x^*, f, \mathbf{u})] - \lambda\right] \le \Pr\left[|\mathrm{sc}(x^*, f, \mathbf{u}) - \mathbb{E}[\mathrm{sc}(x^*, f, \mathbf{u})]\,| \ge \lambda\right] \le \exp\left(-\frac{2\lambda^2}{n}\right) = \frac{1}{\sqrt{n}} \ .$$

Now, observe that for any $x \in A$ and $i \in N$, the score x receives from agent i when applying f on the utility profile u_i is $\alpha_1 = 1$ with probability $u_i(x)$ and α_j with probability $\frac{1-u_i(x)}{m-1}$ for $j = 2, ..., m$. Hence,

$$\begin{aligned}
\mathbb{E}[\mathrm{sc}(x, f, \mathbf{u})] &= \sum_{i \in N} \left(u_i(x) \cdot 1 + \sum_{j=2}^{m} \frac{1 - u_i(x)}{m-1} \cdot \alpha_j \right) \\
&= \sum_{i \in N} \left(u_i(x) + (1 - u_i(x)) \frac{m - d_R - 1}{m-1} \right) \\
&= \frac{m - d_R - 1}{m-1} n + \frac{d_R}{m-1} \mathrm{sw}(x, \mathbf{u}) \ .
\end{aligned}$$

Therefore for every alternative $x \in L$ it holds that

$$\begin{aligned}
\mathbb{E}[\mathrm{sc}(x^*, f, \mathbf{u})] - \lambda &= \frac{m - d_R - 1}{m-1} n + \frac{d_R}{m-1} \mathrm{sw}(x^*, \mathbf{u}) - \lambda \\
&\ge \frac{m - d_R - 1}{m-1} n + \frac{d_R}{m-1} \mathrm{sw}(x, \mathbf{u}) + \lambda \\
&= \mathbb{E}[\mathrm{sc}(x, f, \mathbf{u})] + \lambda \ .
\end{aligned}$$

Using this last observation and the Hoeffding bound for the random variable $\mathrm{sc}(x, f, \mathbf{u})$, we have

$$\begin{aligned}
\Pr\left[\mathrm{sc}(x, f, \mathbf{u}) \ge \mathbb{E}[\mathrm{sc}(x^*, f, \mathbf{u})] - \lambda\right] &\le \Pr\left[\mathrm{sc}(x, f, \mathbf{u}) \ge \mathbb{E}[\mathrm{sc}(x, f, \mathbf{u}) + \lambda\right] \\
&\le \Pr\left[|\mathrm{sc}(x, f, \mathbf{u}) - \mathbb{E}[\mathrm{sc}(x, f, \mathbf{u})]| \ge \lambda\right] \\
&\le \exp\left(-\frac{2\lambda^2}{n}\right) \\
&= \frac{1}{\sqrt{n}} \ .
\end{aligned}$$

By the union bound the probability that some of the undesirable events happen is at most $m/\sqrt{n}$. Hence, with probability at least $1 - m/\sqrt{n}$ there is no $x \in L$ among the winners. We conclude that the expected social welfare of the worst winner is at least

$$\left(\mathrm{sw}(x^*,\mathbf{u}) - \frac{2\lambda(m-1)}{d_R}\right)\left(1-\frac{m}{\sqrt{n}}\right) = \mathrm{sw}(x^*,\mathbf{u})\left(1-\frac{(m-1)\sqrt{n\ln n}}{d_R\cdot \mathrm{sw}(x^*,\mathbf{u})}\right)\left(1-\frac{m}{\sqrt{n}}\right)$$
$$\geq \mathrm{sw}(x^*,\mathbf{u})\left(1-\frac{m(m-1)}{d_R}\sqrt{\frac{\ln n}{n}}\right)\left(1-\frac{m}{\sqrt{n}}\right)$$
$$\geq \mathrm{sw}(x^*,\mathbf{u})\left(1-\frac{m(m-1)}{d_R}\sqrt{\frac{\ln n}{n}}\right)^2$$
$$\geq \mathrm{sw}(x^*,\mathbf{u})\left(1-\frac{2m(m-1)}{d_R}\sqrt{\frac{\ln n}{n}}\right)$$
$$= \mathrm{sw}(x^*,\mathbf{u})\left(1-\epsilon(n,m)\right) \quad .$$

The first transition follows by substituting λ, the second transition holds since $\mathrm{sw}(x^*,\mathbf{u}) \geq n/m$, the third transition easily follows by the condition on $\epsilon(n,m)$ using $n \geq 3$, and the fourth transition follows from $(1-\alpha)^2 \geq 1-2\alpha$. This concludes the theorem's proof. □

As corollaries, we have that if $n/\ln n \geq 16m^2(m-1)^2$, then the distortion of Embedding 5 into Veto is at most two. In addition, for instances with $\epsilon(n,m) = o(1)$, the distortion is at most $1+o(1)$.

When n is not much larger than m we can show an exponential lower bound on the distortion of Embedding 5 into Veto by exploiting the relation to the well-known *coupon collector problem* (see, e.g., [9]). Indeed, consider an instance with n agents with utility 1 for a particular alternative x^* and 0 for all other alternatives. Then, our embedding into Veto should select equiprobably among all alternatives besides x^* the one that will not get score 1 (i.e., the one that will be ranked last). Hence, the question of whether every alternative besides x^* was ranked last in the preferences of some agent is the same as the question of whether all $m-1$ coupons will be randomly selected after n trials. When $n = (m-1)\ln(m-1)/2$ the probability that this happens is exponentially low (i.e., $O(\exp(-\sqrt{n}))$) and, therefore, the expected social welfare of the alternative to be selected will be at most $O(\exp(-\sqrt{n}))$ and the distortion at least $\Omega(\exp(\sqrt{n}))$. More generally we have the following theorem. Note that, in order to keep the exposition simple, we assume that f simply returns the vetoed alternative (as opposed to a ranking of all the alternatives).

Theorem 14. *Let $n = |N| \geq |A| = m$, and let $f : \mathcal{U} \to \Delta(A)$ be a randomized embedding into Veto. Then $dist(f) = \Omega(m/\sqrt{n})$.*

Proof. Let f be a randomized embedding into Veto. Let $N = N' \cup N''$, where $|N'| = n-\lambda$ and $|N''| = \lambda$, with λ to be defined later. We define a utility profile $\mathbf{u} \in \mathcal{U}^n$ as follows. For all $i \in N'$ and $x \in A$, $u_i(x) = 1/m$, that is, all the agents in N' have utility $1/m$ for each alternative. Let $x^* \in A$ be the alternative that has the highest probability of being vetoed under f given this utility profile, i.e., for all $i \in N'$,

$$\Pr[f(u_i) = x^*] \geq \Pr[f(u_i) = x] \quad . \tag{6}$$

Furthermore, for all $i \in N''$ we have $u_i(x^*) = 1$, $u_i(x) = 0$ for all $x \in A \setminus \{x^*\}$. Note that $\mathrm{sw}(x^*,\mathbf{u}) = \lambda + (n-\lambda)/m$, whereas $\mathrm{sw}(x,\mathbf{u}) = (n-\lambda)/m$ for all $x \in A \setminus \{x^*\}$.

It follows from Equation (6) that the the probability that x^* is among the $\lambda+1$ alternatives that are vetoed least by the agents in N' is at most $(\lambda+1)/m$. Therefore with probability at least $1-(\lambda+1)/m$ there are $\lambda+1$ alternatives that are vetoed at most as many times as x^* by N', and at least one of them is not vetoed by the agents in N'' as there are only λ such agents. We conclude that the distortion of f is at least

$$\text{dist}(f,\mathbf{u}) \geq \frac{\lambda+\frac{n-\lambda}{m}}{\frac{\lambda+1}{m}\cdot\left(\lambda+\frac{n-\lambda}{m}\right)+\left(1-\frac{\lambda+1}{m}\right)\cdot\frac{n-\lambda}{m}} = \frac{\lambda+\frac{n-\lambda}{m}}{\frac{\lambda+1}{m}\cdot\lambda+\frac{n-\lambda}{m}} .$$

Taking $\lambda=\Theta(\sqrt{n})$, we get that $\text{dist}(f)\geq\Omega(m/\sqrt{n})$. □

Theorem 14 provides a necessary condition $n=\Omega(m^2)$ in order to obtain constant distortion into Veto, whereas Embedding 1 yields a constant upper bound when $n=\Theta(m\ln m)$ with respect to Plurality and Approval. Hence, randomized embeddings into Veto are provably less efficient even when the number of agents is larger than the number of alternatives.

6 Discussion

In this section we discuss a few prominent issues.

On the interpretation of our results. Interestingly, our technical results deal with embeddings into voting rules and are not directly related to communication, therefore the results may lend themselves to different interpretations.

Procaccia and Rosenschein [11] motivate their work by arguing that in some systems voting must be used since utilities cannot be calculated or cannot be compared, even though exact and comparable utilities conceivably *exist*. This situation is less common in systems that are populated entirely by computational agents (although bounded rationality may be an issue), and more common in systems that also involve humans. Procaccia and Rosenschein do assume that the agents rank the alternatives according to decreasing utility, but this does not require calculating the exact utilities.

In contrast, the randomized embeddings introduced in this paper require the computation of exact utilities. So, we are dealing with systems (that are presumably populated solely by computational agents) where one could potentially gather exact utilities and select the best alternative. One possible interpretation of our results is that they can be used to improve the performance of systems that, by design, are unnecessarily constrained to use voting when utilities could have been reported, but this is not an approach we wish to advocate. We feel that our interpretation in terms of communication reduction, as presented in the introduction, provides a robust motivation that ties in closely to the results.

Relation to work on compact preferences. A significant body of work in AI is devoted to compactly representing preferences. A prominent example is the work on *CP-nets* [2], a graphical representation of preferences that employs conditional *ceteris paribus* (all else being equal) preference statements. This representation is often compact and admits efficient algorithms for different inference tasks. Another example is a recently proposed representation of utility functions using weighted propositional formulas [15]. By considering different restrictions on the syntax of formulas and the weights one can obtain different representation languages, each capturing a different class of utility functions.

This line of work proposes to reduce communication and computation burdens by making arguably natural assumptions regarding the utility functions. In contrast, in this paper we impose no restrictions on the utility functions (with the exception of the extremely weak normalization assumption, see below); rather, we reduce communication by slightly relaxing the optimality of the outcome.

Generality of normalized utilities. It is easy to see that very strong lower bounds would hold without assuming that the utilities are normalized. We argue though that this assumption is essentially without loss of generality. Indeed, the setting we have in mind (which is consistent with our motivation and examples) is one where all agents have equal weight in deciding the social quality of alternatives, and are merely trying to evaluate which alternative is best for the system. For example, we are precluding a situation where one agent has utility one for x and zero for the rest, and a second agent has utility $1/2$ for x and zero for the rest: if both believe that x is the only reasonable alternative, they would both give that alternative their entire "pool of points". In other words, the only real assumption is that agents have equal weight; normalized utilities logically follow. Note that a similar assumption is typically made in social choice contexts concerning the fair division of a good among agents with cardinal utilities over parts of the good (see, e.g., [3]); it is assumed that the sum of an agent's utilities for every partition of the good is one.

Future work. Our notion of embeddings into voting rules is extremely decentralized, that is, the agents cast their votes independently according to the embedding. On the other extreme, if full coordination is allowed, the distortion would always be one, as the agents would be able to find out which alternative maximizes social welfare and coordinate their votes in a way that this alternative is elected (assuming the voting rule is onto the set of alternatives). It would be interesting to investigate a notion of embedding that allows for partial communication between the agents.

Our strongest positive results hold in settings where the number of agents is larger than the number of alternatives. This is indeed the case in many environments, notably in political elections. However, one can think of a variety of multiagent settings where the number of alternatives is larger. Our experimental results shed some light to the distortion of randomized embeddings into Plurality and Approval in this case. In future work, we would like to achieve a better understanding of the achievable distortion when $n = o(m)$.

The results in this paper mainly concern three voting rules: Plurality, Approval, and Veto. Certainly, low distortion can also be achieved using randomized embeddings into any scoring protocol as Theorem 12 indicates. Among others, this includes k-approval and k-antiapproval voting rules, according to which each agent approves or vetoes k alternatives, respectively; these voting rules require $O(k \log m)$ bits to be communicated by each agent and meet the low communication requirement when k is small. Furthermore, it includes the Borda rule which, in our context, can be defined as follows: each agent awards $1 - \frac{k-1}{m-1}$ points to the alternative ranked in the k'th position. Extending this line of work to other prominent voting that either have low communication requirements, such as Plurality with Runoff or Single Transferable Vote [4], or are based on pairwise comparisons, such as Copeland or Maximin, could prove challenging but rewarding.

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