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THE EFFECT OF TOOLS OF A COMPUTER MICROWORLD ON STUDENTS' STRATEGIES REGARDING THE CONCEPT OF CONSERVATION OF AREA

RUNNING HEAD : CONSERVATION OF AREA TOOLS AND STUDENTS' STRATEGIES

ABSTRACT. This study focuses on the role of tools, provided by a computer microworld, (C.AR.ME) on the strategies developed by 14-year-old students regarding the concept of conservation of area of a non-convex polygon. Students' strategies on a transformation and a comparison task were interpreted and classified into categories in terms of the tools used for the development of these tasks. The analysis of the data shows that the nature of the tools used affected the nature of solution strategies that the students constructed. Three different approaches to the concept of conservation of area emerged from the strategies which were constructed by the students in this microworld: the intuitive approach involving the splitting of areas into parts and recomposing them to produce equivalent areas, the enclosing of the non-convex polygon in a minimum convex superset and the dynamic transformation approach. Most students managed to use the above approaches in combination thereby viewing the concept of conservation of area as interrelated with the concept of area measurement using spatial units and area formulae. Almost all students experienced qualitative aspects of the conservation of area through being involved in the process of splitting areas into parts and recomposing them to produce equivalent areas. Most students experienced dynamic representations of this concept through exploring it in a variety of equivalent areas. Moreover, most students explored the conservation of area in classes of equivalent parallelograms and triangles and this illuminated serious difficulties, most of which were overcome in this computer environment. Finally, the analysis of the data shows that all students were involved in the tasks and succeeded in completing them with more than one correct solution strategy thereby developing a broader view of the concept, although not all of them used the same strategies.

KEY WORDS : conservation of area, computer microworld, computer tools, secondary education, educational research, geometry, problem solving

1. INTRODUCTION

Conservation of area is a fundamental and preliminary aspect in students' understanding of the concept of area measurement (Piaget, Inhelder & Szeminska, 1981; Hirstein, Lamb & Osborne, 1978; Maher & Beattys, 1986). Area measurement is part of mathematics, science and technology but also of everyday life (Hirstein, Lamb and Osborn, 1978; Sanders, 1976). It is also closely related to the number concept (Skemp, 1986). Conservation means that the quantitative value of an area remains unaltered while its figure can be qualitatively new (Piaget et al., 1981). Understanding this concept is a process of giving meaning to its different representations: for example numerical, visual and symbolic. Students have the opportunity of expressing their own knowledge of the above concept by selecting from among representations those most appropriate to their cognitive development and of constructing a broader and more abstract view of this concept by selecting more than one representation system. Moreover, the meanings that students can give to the concept of the conservation of area is closely related to the tools that they use and to the shapes that they have to study. Previous research literature has been based on Piaget's work (Piaget et al., 1981) and has investigated students' thinking on the concept of conservation of area as a preparatory concept of area measurement. This view is reflected in the tools proposed to the students by emphasizing the use of paper and scissors in combination with the students' sensory - motor actions to cut, move and paste the parts of a shape to recompose a new equivalent one (Hiebert, 1981; Liebeck, 1987; Rahim, & Sawada, 1990). The concept of conservation of area has been investigated in isolation from the concept of area measurement and of area formulae. Understanding all these concepts in integration is a complicated but also an essential process for the students at early ages as well as later in secondary school.

Individual shapes of different form have been used to investigate students' thinking on the concept of conservation of area, leaving out the classes of equivalent shapes of the same form. Examples of these classes are equivalent triangles or parallelograms with common bases and equal heights. While the number of the classes above can be innumerable as well as the number of shapes included in each individual class, students' thinking about the concept of conservation of area in these classes of shapes has not yet been investigated by other researchers. The above classes of shapes can be viewed as dynamic visual representations of the concept of conservation of area. By understanding the invariance of the area of these shapes students can extend their meanings about this concept, giving it a dynamic character. In addition, students' thinking on conservation of area in relation to a non-convex (concave) polygon has not been yet reported in the literature.

In the school context students are introduced early to the use of area formulae but the concept of conservation of area is overlooked. Despite the fact that students can achieve the concept by studying it in a variety of shapes and using a number of different tools; students do not have this opportunity in their schools. The tools that are usually used, do not support the construction of a variety of representations of the concept. Moreover, the shapes that they are invited to study are a small part of the plethora of forms which could be studied.

Although the research literature has explored the possibility of the computer in providing students with the opportunity to experience multiple or dynamic representations of a concept, the case of the conservation of area has not been investigated. In an attempt to create an environment for the students to experience different representations of the conservation of area, the ‘Conservation of ARea and its MEasurement’ microworld (C.AR.ME) has been developed (Kordaki & Potari, 1998a). This microworld is an open problem solving exploratory environment which offers tools to help students construct multiple representations of the concept of conservation of area and its measurement. In this paper, the effect of the tools of this microworld on students’ constructions is investigated. More specifically, I explore the following :

- students' strategies regarding the concept of conservation of area and their development while interacting in the context of the computer microworld;
- students’ thinking on the concept of conservation of area in classes of equivalent triangles and parallelograms with common bases and equal heights;
- the role of the tools that are offered by the computer microworld regarding students' strategies.

2. THE CONCEPT OF CONSERVATION OF AREA

Area as a space inside a figure and the concept of conservation of area are preliminary concepts for the understanding of the concept of area measurement (Hughes & Rogers, 1979; Piaget, et al., 1981; Beattys & Maher, 1985). Area is a stable attribute - a definite measurable size of the plane surfaces enclosed by figures (Piaget et al., 1981; Douady & Perrin, 1986). An area may be conserved while the shape of its figure is altered. Conservation of area means that a whole area - which is made up of sub-areas organized in one way - can remain invariant in spite of rearrangement of its parts (Piaget et al., 1981). This rearrangement implies conservation -both of parts and of wholes. The ability to analyze a whole area in this way is prerequisite to area measurement because when measuring an area we assume, as we do for all measurement, that partial units are conserved and can be composed in a variety of ways to form invariant wholes (Piaget, et al., 1981, p. 262). Basic aspects in the understanding of the concept of conservation of area are the concept of compensation and the part - whole aspect (Steffe & Hirstein, 1976; Piaget, et al., 1981; Carpenter & Lewis, 1976). Reversibility and transitivity also imply conservation and are implied by it (Piaget, et al., 1981; Steffe, 1971). Students can master these concepts through the *cut, move* and *paste* actions in re-arranging the parts of a shape to produce a new one with equivalent area. Understanding the concept of conservation through these processes is necessary and prerequisite to students understanding the concept of area measurement as well as multiplication structures (Hirstein et al., 1978; Douady & Perrin, 1986).

Students have difficulties in understanding the possibility of equivalence of an area when it is represented in shapes of different forms (Carpenter et al., 1975). Moreover, students have problems understanding an area as the sum of its parts (Brown, Carpenter, Kouba, Lindquist, Silver & Swafford, 1988). In addition, they are prevented from understanding the concept of conservation of area since their conclusions are based on their perceptions. They are also reluctant to compare areas because they focus on the shapes' most dominant dimensions (Carpenter, 1976; Hughes & Rogers, 1979). In making conclusions students cannot relate numerical information to visual information if it does not match (Carpenter, 1976). Moreover, students confuse areas and their perimeters and use them alternatively (Hart, 1984; 1989; Kidman & Cooper, 1997). Hence, they conserve areas by conserving their perimeters and vice versa (Hart, 1989). Difficulties in understanding the concept of conservation also arise with the concept of the unit of area (Hart, 1989). Students cannot conserve units from their parts. In some cases they count only the whole units while in others they count the parts which are greater than half of a unit as whole units neglecting the remaining parts of the units (Carpenter et al., 1975).

The understanding of the concept of conservation of area is also related to the form of the shapes to be conserved. Most of the research literature emphasizes studying students' difficulties with standard geometrical shapes such as squares, rectangles, parallelograms and triangles (Johnson, 1986). However, students may understand the possibility of conservation of area in squares and in parallelograms but they face difficulties in understanding this concept in triangles (Hughes & Rogers, 1979). In particular in the case of equivalent rectangles students cannot realize the inverse relationship between their dimensions when area is conserved (Carpenter & Lewis, 1976). Difficulties also appear when students study irregular shapes (Maher & Beattys, 1986). In these shapes students appear to lose the fundamental concepts of the conservation of area and the unit of area measurement (Maher & Beattys, 1986; Liebeck, 1987). Students' difficulties about the concept of conservation of area are not easily overcome through age transition. Sometimes these difficulties remain during students' studies in secondary school or until they become adults. Moreover, it has been observed that these difficulties still remain with prospective student teachers (Maher & Beattys, 1986; Hart, 1989; Tierney, Boyd & Davis, 1986).

Despite the importance of the concept of conservation of area the study of this is not emphasized in the school curriculum in primary or secondary school. In primary school students are immediately introduced to the operation of area measurement. The square unit is taught for use and subsequently they are directed rapidly to the use of area formulae.

Students' difficulties regarding area measurement are related to their premature introduction to the quantitative approach to area using area formulae, while disregarding the qualitative approach which emphasizes the concept of conservation of area without the use of numbers (Johnson, 1986; Douady & Perrin, 1986; Hirstein et al., 1978). The qualitative approach acknowledges the need for students to grasp the concept of conservation of area by splitting areas into parts or units and recomposing them in order to construct new equivalent shapes (Liebeck, 1987; Carpenter et al., 1975; Hiebert, 1981). Moreover, students' difficulties in area measurement are attributed, firstly, to a fragmented way of studying areas without dynamic relation to their perimeters (Moreira Baltar, (1997) and secondly, to the inability to fill the gap between the cultural approach to area, expressed in the use of area formulae, and the qualitative approach to manipulating areas without the use of numbers (Baturo & Nason, 1996).

3. THE COMPUTER MICROWORLD

The 'Conservation of Area and its Measurement' (C.AR.ME.) microworld (Kordaki & Potari, 1998a) has been designed as an interactive, open, problem solving and exploratory environment to support students in constructing actively their own approaches to the concepts of conservation of area and its measurement. In this context a variety of tasks can be posed by the teacher and a number of different representations of the above concepts can be constructed or explored by the students. The microworld intends to encourage students to be responsible for their learning process, by not providing the 'right' answers but by offering intrinsic visual feedback to their actions.

For the design of this microworld three models were constructed : a model of learning, a model of subject matter and a model of students' possible actions when they are involved in the conservation and measurement of area. The model of learning is viewed from a constructivist perspective (von Glasersfeld, 1990) as an active, subjective and constructive process. Social views were also taken into consideration concerning the role of tools in the learning process (Crawford, 1996a, 1996b; Noss and Hoyles, 1996; Vygotsky, 1978). The model of the subject matter is based on an analysis of the fundamental aspects that constitute the concepts of conservation of area and its measurement as has been reported in previous research. The model of students' possible actions is based on students' sensory-motor actions. These actions are reported in the literature as essential for the understanding of these concepts. The above theoretical considerations were interpreted in operations and specifications of tools that were provided for the pupils. The three models, the general design of C.AR.ME. and the provided tools are further discussed in Kordaki & Potari (1998a). The provided tools as they are presented in the interface of C.AR.ME are illustrated in Figure 1.

Figure 1 about here

In the context of C.AR.ME. a variety of tools were provided to help the students to express their own knowledge by selecting among them the most appropriate to their cognitive development. These tools also supported each student in constructing a variety of representations of the concepts of conservation of area and its measurement. Some of these representations have a qualitative character, some others a quantitative one while others are exploratory and dynamic. By interacting with different representations, students have the opportunity to construct bridges between different approaches of these concepts. Regarding the concept of conservation of area two groups of tools are provided for the students :

a) Tools that simulate the students' sensory-motor actions. These tools can be used for conserving areas by manipulating them without the use of numbers. A variety of different representations of equivalent areas can be created in two ways: first, by changing only the position of a figure while conserving its shape and second, by splitting a figure into its non overlapping parts and recomposing the parts to form a new equivalent shape. These processes can be realized by using the tools of '*Select All'*, '*Select Part'*, '*Cut*', '*Paste*', '*Rotate*', and '*Symmetry*' (These tools are presented under the 'Edit' column, in Figure 1). The above tools were designed to help students to study the concept of conservation of area in a qualitative way. Examples of the use of these tools are presented in Figure 2.

Figure 2 about here

b) Tools for automatic transformations. A number of different tools (presented under the 'Automatic Transformations' column, in Figure 1) are provided for the students to automatically transform areas already drawn, to equivalent ones. These equivalent areas are a *square*, a *rectangle with dimensions of ratio 1:2, a right-angled triangle with perpendicular sides of ratio 1:2*, and *classes of equivalent shapes* of the same form such as *rectangles, parallelograms and triangles* with common bases and equal heights (Figure 3). The above tools were designed to help students to study the concept of conservation of area in a dynamic way.

Figure 3 about here

Students can draw the base of a representative for each one of the above classes using the drawing tools of C.AR.ME. Then, students can produce a number of equivalent shapes belonging to each class by using the appropriate tools. By altering these bases, a number of different classes of equivalent shapes of the same form can be produced.

In the C.AR.ME. microworld two sets of tools for area measurement were also provided for the students :

a) Tools to construct a variety of spatial units and grids. A variety of units and grids are offered by the microworld as tools for measurement to the students. These are: *a square unit, a rectangular unit*, the corresponding *grids*, and a *triangular grid*. Two additional tools are available for the students to create *personal units* and *grids*. Moreover, tools to perform *unit iteration* and *counting the number of units* needed to cover specific areas are also available (All these tools are presented under the 'Measurement Tools' column, in Figure 1). Students have the opportunity to experience the concept of conservation of area in a more sophisticated way by splitting areas using spatial units, counting and recomposing them to construct new equivalent shapes.

b) A tool for '*automatic area measurement*' using the standards units of area. By using the above tool (This tool is presented under the 'Automatic Measurements' column, in Figure 1). students have the chance to investigate the equivalence of different areas or to verify the equivalence of known equivalent ones. A more detailed description of the operations of the microworld is presented in Kordaki & Potari, (1998a).

4. THE CONTEXT OF THE STUDY

4.1. The theoretical framework and the focus of the study

This study focuses on students' approaches (in the sense of students' conceptions) to the concept of conservation of area used in their strategies in solving the problems of transformation and of comparison within the context of the C.AR.ME. microworld.

This work is part of an extensive formative evaluation of the above microworld designed to investigate students' strategies related to the concepts of conservation of area and its measurement (Kordaki,1999). The part of the research regarding students' strategies in the concept of area measurement is reported in Kordaki & Potari, (2002). This evaluation study was realized in the field with students and focused on their learning processes and not only on the learning outcomes. In terms of methodology it is a qualitative study (Cohen & Manion, 1989) as well as adopting a phenomenographic approach to evaluation (Marton, 1988). In interpreting this approach I focus on the variety of relations/interactions realized by the students with the tools of C.AR.ME. as well as on the different ways that these students approach the concept of conservation of area using these tools.

4.2. The Learning experiment

The learning experiment took place in a typical state secondary school of Patras, Greece. A complete class consisting of thirty 2nd grade students (14-year-old) participated in a problem-solving activity. This activity involved the conservation of area and its measurement concepts and was not part of students' normal classroom experience. In this experiment students were asked to face two tasks in the context of C.AR.ME. The duration of each task was commensurate with the students' needs. Each student spent on average about two hours per task. A familiarization phase using the tools of C.AR.ME. took place before the students commenced the main study. The aim was to familiarize the students with the tools of the microworld and not to get them involved in the solving of specific task processes. The need for this phase emerged from the pilot study (Kordaki & Potari, 1998b) and was realized by asking the students to try consecutively all the operations provided by this microworld. For example, ‘draw a polygon’, ‘draw a segment’, ‘save your work’, ‘rotate a shape’, ‘select the rectangular grid’, ‘copy a shape’, ‘paste a shape’ were some typical examples of the tasks posed.

Students worked in a computer laboratory, consisting of three computers where they worked in rotation. They worked individually except during the familiarization phase where they worked in pairs. The researcher participated in the study as an observer with minimum intervention. All interventions realized by the researcher are reported with reference to the specific cases in the results section of this paper. The whole learning context consisted of the computer environment, the researcher, the students and the tasks they performed during the experiment.

The data resources are the log files containing the history of students' interactions with the tools of C.AR.ME., the electronic snapshots of students drawings, the audio recordings of all verbal interactions and the field notes of the researcher.

4.3. The tasks

Two tasks were assigned to the students during this evaluation study. The first was the task of transformation of a non-convex polygon to another polygon with equal area and the second was the comparison of a non-convex polygon to a square, not easily comparable by ‘eye’. All the shapes to be studied were drawn on the computer screen by the researcher, and the instructions were presented verbally to the students. In the first task students were asked to: ‘transform this polygon into another polygon with equal area in any possible way’. Those students who automatically transformed the non-convex polygon into other equivalent geometrical shapes or classes of shapes of the same form were asked: a) ‘what do you think about the areas of these shapes?’, ‘are they equal or not?’, ‘justify your answer”. These questions were asked to give students the opportunity to express their thinking about the concept of conservation of area in classes of equivalent parallelograms and of triangles with common bases and equal heights. For those students who did not use these automatic transformations, the researcher intervened by performing them. Then these students were also asked the questions mentioned above.

By allowing and asking for a variety of solutions to the given tasks students were encouraged to construct their own individual approaches to the concepts of conservation of area and its measurement and to express different areas of knowledge they possess regarding these concepts (Weir, 1992; Lemerise, 1992). The above tasks have been considered by other researchers as essential for the students to construct in a qualitative way the concepts of conservation of area and its measurement (Carpenter, Coburn, Reys & Wilson 1975; Hiebert, 1981; Driscoll, 1981). The nature of these tasks allows the students to construct their own individual approaches to the relative concepts, to express their intuitive knowledge as well as to develop multiple and different solution strategies (Kordaki & Potari, 2002**).**

4.4. The process of analyzing the data

The various types of data were organized according to the two different tasks. In each task all individual students’ multiple-solution strategies were identified and reported. These strategies were analyzed in terms of students’ conceptions of the concept of conservation of area and their development as they emerged during the experiment. In the next stage the focus was on the entire group of students and a classification of strategies into categories was constructed. The criterion for this classification was the kind of tools used by the students to construct each strategy. Moreover, the students’ strategies regarding the concept of conservation of area across categories were discussed as well as the role of the provided tools in the construction of these strategies.

5. RESULTS

Students constructed 328 strategies to complete the tasks of transformation and of comparison in the context of C.AR.ME. These strategies were classified into 28 categories with the criterion being the tools used by the students and presented in Table I (Table I).

Table 1 about here

In this way 14 categories were formed for each task. Twenty of these categories: the categories G1, G2,…G9 and C1,..C11, are different as is shown in Table I. In this table the number of students who performed strategies in each category is also presented. Of the above strategies, 154 were studied in relation to the concept of area measurement and were reported in Kordaki & Potari, (2002). These strategies were classified in eleven categories namely C1, C2,…C11. The remaining 174 strategies are presented and discussed in this article in relation to the concept of conservation of area. These 174 strategies were classified in nine categories (G1,… ,G9). In the following section 5.1 the categories of students' strategies regarding the concept of conservation of area are presented and discussed and in section 5.2. the students' strategies across these categories.

5.1.Categories of students' strategies

The students' strategies are presented as they were performed. Their difficulties are discussed in reference to the specific cases that appeared. The fact that most of these strategies were correct is possibly due to their having the opportunity to select among various tools those that allowed them to express their own personal knowledge. The strategies that fall in each category are discussed in relation to those students who performed them.

G1: Transforming and comparing areas by using the tools for automatic transformations. Students transformed the non-convex polygon by performing the following strategies (fig. 4).

Figure 4 about here

Most of the students (28) transformed automatically the non-convex polygon to a number of equivalent shapes (Figure 3). All of these students recognized that the automatically produced shapes could be equivalent, that is, have the same area. To justify their opinion some of them stated that '*all these shapes have the same area because they are produced as equivalent by the computer'* while other students started to develop methods to compare these shapes. The developed methods were a) the automatic measurement of area of the shapes under discussion b) the superimposition of these shapes by using the *copy*, *past*, *rotation* and *symmetry* operations of the C.AR.ME microworld. Regarding the equivalence of the area of the families of equivalent parallelograms students (24 students) who performed this strategy stated that '*these shapes could be equivalent*'. Most of these students (15) justified their opinion by saying that '*any new parallelogram in the family seems to* *be produced by the previous one in the following way : by subtracting an area of triangular form from the left area of the existing parallelogram and adding this triangle-area to its right side, a compensation of area is exercised'*. Another group of five (5) students automatically measured different parallelograms to investigate their equivalence. Four (4) students compared them by superimposing them and manipulating their non common parts using the tools 'cut', 'past', 'rotate' and 'symmetry'. Only four students could recognize that the parallelograms included in the previously referred to family have common bases and equal heights, and as a result they concluded that these parallelograms would have equal areas if the calculation formulae were used.

Regarding the equivalence of the area of the families of equivalent triangles most students who performed this strategy (22 students) stated that '*these shapes can't be equivalent'*. They justified their opinion by focusing on the perimeter of the triangles and saying that '*this triangle can't have the same area as the others because they do not have equal sides'*, '*this triangle can't have the same area as the other because it has a larger perimeter'*. Here, the researcher led these students to a cognitive conflict by asking them to think 'why does the computer operation state that these families of triangles are equivalent'. Then some students (15) automatically measured the area and the perimeter of a number of different triangles and inferred that '*it is unbelievable!!, these triangles have different perimeters but they have equal areas'*, '*I'm surprised!!, I never realized that two triangles could have equal areas even though they have different forms'*, '*Today I learned that two triangles could have equal areas even though they are not congruent'*. Five (5) students superimposed different triangles to compare their areas. However, only three students who realized this strategy were able to recognize that the common properties of the triangles included in the above equivalent families are their common bases and equal heights. Then, these students recognized the conservation of area in these shapes using the appropriate area formulae.

Most students performed these transformations as their first strategy to perform the task of transformation. This indicates a preference for fast and easy transformations automatically performed.

Regarding the task of comparison students (10 students) performed strategies presented in fig 5.

Figure 5 about here

Strategies presented in figure 5 indicate students' preference for comparing shapes of the same form as well as for transforming areas so that their comparisons could be visually performed. The visual comparison is possible as the automatically produced equivalent shapes of the same form are superimposed automatically in such a way that one of their right angles is superimposed. Examples of the above strategies are presented in fig 6.

Figure 6 about here

As this comparison task followed the task of transformation, students who performed the above strategies did not again ask to investigate the equivalence of the automatically produced shapes. Moreover, students who performed the above strategies seemed to understand the concept of transitivity in the process of comparison of areas.

G2: *Transforming and comparing areas by using the tools that simulated the students' sensory-motor actions.* Most students (27 students) transformed the non-convex polygon by performing strategies included in this category. These strategies are presented in fig 7 :

Figure 7 about here

The solution strategies in this category indicate an attempt by the students to express their intuitive understanding of the concept of conservation of area in this computer environment. By performing these strategies, students (19 students) viewed that an entire area can be conserved after a number of basic area transformations (subcategory G2a). These basic transformations constituted the parallel translation of area, the rotation of a figure through an angle about a point of rotation, the construction of the reflection of a figure about an axis of symmetry as well as combinations of these transformations. Moreover, students (16 students) viewed the concept of conservation of area as a combination of the conservation of its parts (subcategory G2b). Examples of students' constructions have been shown in Figure 2a. Students conserved the parts, performing on them a sequence of the previously referred to basic area transformations. All these activities were performed by these students, using the tools that simulated the students' sensory-motor actions, splitting an area (e.g. the non-convex polygon) into its parts and recomposing them, producing new equivalent areas. To realize successfully the whole previously described transformation activity, students had to plan a sequence of basic area transformations. Students had to act in a conscious way to perform the above activity in this computer environment despite the fact that they usually perform the same task acting in an instinctive way in the paper and scissors environment. Students also had the opportunity to experience in a visual way, the mathematical concepts of parallel translation, rotation and symmetry which are implied in the basic transformations. These concepts were unknown to these students until this time. More specifically, during the task of comparison students who performed the related strategies (fig.8), estimated visually the correct position of an axis of symmetry so that the reflection of the square could be superimposed onto the non-convex polygon (Fig. 2b). Moreover, other students estimated visually the appropriate size of the angle of rotation and the point of rotation so that the non-convex polygon could be superimposed onto the square after a sequence of rotations. Here, most of these students faced a number of difficulties as it was very hard for them to foresee and plan the appropriate actions so as to succeed in the superimposition process using the rotation tools of C.AR.ME. Some students overcame the above difficulties by trial and error.

Students comparison strategies are presented in figure 8 :

Figure 8 about here

By performing the comparison strategies presented in the above figure students (14 students) seemed to express an understanding of the concept of transitivity in the area comparison process. These students also expressed their intuitive knowledge about the concept of conservation of area and used this knowledge to compare visually the areas under study.

G3. *Transforming and comparing areas by using the tools that simulate the students' sensory-motor actions in combination with the tools for automatic transformations.* Students' transformation strategies which fall in this category are presented in fig. 9 :

Figure 9 about here

The first two strategies in figure 9 can be viewed as a sequence of two transformations while the rest can be viewed as a sequence of three. Examples of students' strategies are presented in Fig. 10.

Figure 10 about here

Students also used the above tools to perform the task of comparison. The performed strategies are presented in fig 11.

Figure 11 about here

The comparison strategies presented in figure 11 seemed to be an attempt by the students who performed them (16 students), to transform the areas under study so as to compare areas of the same form. All this transformation process led these students to making visual conclusions about the comparison of the areas.

Students' strategies in this category intimate an understanding of the concept of conservation of area as a combination of different approaches to conservation through a sequence of transformations. The combination of conservation involves the exploratory and intuitive approaches described in the categories G1 and G2 respectively. This combination of conservation also implied an understanding of the concept of transitivity in the comparison of areas.

G4. Transforming areas by using the tools that support the operation of area measurement using spatial units in combination with the tools for automatic transformations. One student performed the following transformation strategy (fig. 12)

Figure 12 about here

The student's strategy in this category indicates an understanding of the concept of conservation of area as a combination of different approaches to this concept through a sequence of transformations. First, he/she split the area of the non-convex polygon into area-units and recomposed them to produce a new equivalent square. Secondly, he/she automatically transformed the area of this square into a right-angled isosceles triangle and thirdly he/she automatically transformed the area of this triangle into an equivalent rectangle.

G5. *Transforming areas by enclosing the non-convex polygon in its minimum convex super set using the drawing tools in combination with tools that simulate the students' sensory-motor actions.* One student performed the transformation strategy which is presented in fig. 13.

Figure 13 about here

The student's strategy in this category indicates an intuitive understanding of the conservation of area of the non-convex polygon as a combination of conservation of two convex polygons. The first polygon is the minimum convex superset of the non-convex polygon and the second is the supplement of this polygon in relation to the previously referred to superset. The student's example of this strategy is presented in fig. 14a (Figure 14).

Figure 14 about here

G6. *Transforming areas by enclosing the non-convex polygon in its minimum convex super set using the drawing tools in combination with the simulations of students sensory-motor actions and the tools for automatic transformations.* Two students performed the transformation which are presented in fig. 15.

Figure 15 about here

The students' strategies in this category can be viewed as a combination of two approaches to the concept of conservation. First, the intuitive approach referred to in category G5 and second, a dynamic approach to this concept realized by the automatic transformation of the minimum superset of the non-convex polygon into an equivalent square. The students' example of this strategy is presented in fig. 14b (Fig. 14).

G7. C*omparing areas by enclosing the non-convex polygon in a minimum rectangular/square super set in combination with the operation of area measurement using spatial units.* Two students performed the comparison strategies which are presented in fig. 16.

Figure 16 about here

The strategies described in figure 16 can be interpreted as a combination of two approaches to the concept of conservation. First, the intuitive approach, described in category G5 and second, the approach that emphasizes the conservation of area by splitting it into area-units. Students' example of this strategy is presented in fig. 14c (Fig. 14).

G8. *Transforming and comparing areas by enclosing the non-convex polygon in a minimum convex super set using the drawing tools in combination with the operation of area measurement using spatial units and area formulae.* Four students performed the strategies presented in fig. 17.

Figure 17 about here

Here, as well, students' strategies can be interpreted as a combination of three approaches to the concept of conservation. First, the intuitive approach, described in category G5, second, the approach that emphasizes the conservation of area by splitting it into area-units and third the prepositional approach that stresses the use of area formulae. Students' example of this strategy is presented in fig. 14d (Fig. 14).

G9. *Transforming areas by using the automatic area measurement operation in combination with the area formulae and automatic transformations.* Five students performed the strategy presented in fig. 18:

Figure 18 about here

Here, as well, students who performed this strategy expressed an understanding of the concept of conservation of area as a combination of different approaches to this concept: a conservation of area by a transformation using area formulae and a conservation by a transformation automatically performed by the system. This strategy also implies an understanding of the conservation of area as the union of its non-overlapping subsets. By performing this strategy these students viewed the concept of conservation and the area formulae in integration. In this way these students combined their school knowledge of area formulae with automatic transformations.

5.2. Students' strategies across categories

In Table II, the categories of strategies regarding the concept of conservation of area for each task are presented in terms of each individual student. The number in each cell indicates the order of the performance of the specific strategy. So, a sequence of numbers corresponds to each student. The missing numbers, which can be observed in some sequences, refer to strategies concerning the concept of area measurement.

Table II about here

For example, student P10 performed his first transformation strategy by using the tools for automatic transformation, his second by using the tools that simulate students' sensory-motor actions to conserve an area altering only its position, his forth and sixth strategy by using the tools that simulate students' sensory-motor actions to split areas in parts and recompose them to produce equivalent shapes, and his seventh by using the tools that simulate students' sensory-motor actions in combination with the tools for automatic transformation of area. The first strategy in both tasks and the third strategy in the comparison task were in the same category (G1). The last column of the table shows the total number of strategies related to the concept of conservation of area per student. The last row of the table shows the number of students who performed strategies that fall in each category. The penultimate row presents the number of strategies for each category.

As it is shown in Table II the most common strategies performed by the students were; first, the automatic transformation of area (28 students performed 28 transformation strategies and 10 students performed 15 comparison strategies), and second, the transformation of area using the students' sensory motor actions without the use of numbers (27 students performed 67 transformation strategies and 14 students performed 19 comparison strategies). The first strategy seems to indicate a preference of the students for fast and easily delivered strategies while the second shows that the students still need to be engaged in the process of manipulating areas without the use of numbers despite the fact that students have been introduced to area formulae from primary school. The number of strategies is different from the number of students who performed them as each student performed more than one strategy in each category.

Regarding the order of the appearance of the strategies during the transformation task about half of the students (16) conserved the area of the non-convex polygon by first transforming it automatically into other equivalent typical geometrical shapes (category G1) while four students (4) used this strategy as their second choice. The tools that simulated students' sensory motor actions to conserve areas without the use of numbers were used as a second strategy (category G2) by fifteen (15) students while for eight students (8) it was their first. Both of the above strategies (categories G1 and G2) were used by fourteen students (P1, P3, P4, P5, P7, P8, P10, P11, P12, P16, P18, P22, P28, P30) as their first pair of strategies, while students P11 and P18 did not develop any other solution strategies for the transformation task. The above pair of strategies was realized by the students P5, P20, P21, P22 and P24.

Twenty students (P3, P4, P5, P6, P8 P9, P10, P11, P14, P15, P17, P19, P20, P23, P24, P25, P26, P27, P29 and P30) constructed strategies integrating different areas of knowledge regarding the concept of conservation of area (strategies in categories G3, G4, G5, G6, G7, G8, and G9). These strategies were developed mainly later on during the process of students' involvement in both tasks and show students' mathematical progression regarding the concept of conservation of area to a more advanced level. In my view, this fact shows, that the nature of these tasks in which students are asked to solve them 'in any possible way' and the availability of a variety of tools that support the construction of different solution strategies of the same problem, can inspire the students to move to more advanced mathematical constructions regarding the above concept.

6. DISCUSSION

The concept of the conservation of area, and its approaches referred to in the literature, emphasize the role of paper and scissors to split areas into parts and recompose them to produce equivalent areas (Hiebert, 1981; Liebeck, 1987; Rahim, & Sawada, 1990). In contrast to this one-sided view, a number of different approaches to the above concept were constructed by the students in this computer environment. These approaches were constructed by taking advantage of the variety of different tools provided by the C.AR.ME. microworld in integration with students' school knowledge –area formulae. Moreover, each individual student exploited this variety of tools and constructed a number of different approaches to this concept. In this section are discussed: a) the variety of different approaches to the concept of conservation of area constructed by the students in the context of C.AR.ME. and b) the role of tools in students' strategies.

a) Students' approaches to the concept of conservation of area in the context of C.AR.ME.

The ' intuitive' approach. This approach (reported in category G2) expresses a spatial, qualitative and intuitive understanding of this concept. Despite the fact that this approach can be performed in an unconscious way in the paper and scissors environment, it was not true in this computer microworld. Here, students performed the basic area transformations namely: parallel translation, rotation through an angle about a point of rotation, reflection about an axis and a combination of these, on the entire area of the non-convex polygon as well as on its non overlapping parts. To do this activity, students were forced to make a plan of a sequence of these transformations and to use the specific tools carefully and consciously trying to foresee the results. Here, students had the opportunity to experience visually the previously referred to basic area transformations as well as to compare visually the areas under study. The understanding of the concept of transitivity is also implied in these comparisons.

The ‘enclosing in a minimum convex superset’ approach. Here, students viewed the conservation of area of the non-convex polygon as a sequence of conservations. First, they viewed the conservation of the area of this polygon as a subtraction of two convex polygons. The first polygon is a minimum convex superset of the non-convex polygon and the second polygon is the supplement of the non-convex polygon in relation to the previously referred to superset. For their minimum superset, students constructed the ‘minimum convex closure’ (reported in categories G5 and G6) ‘the minimum square’ (reported in category G8) and ‘the minimum rectangle’ enclosing the non-convex polygon (reported in categories G7 and G8). Second, students viewed the conservation of the area of the non-convex polygon as a synthesis of conservation of the two previously referred to polygons (reported in categories G5, G6, G7 and G8). Students conserved the area of these two polygons in three ways: a) by parallel translation of both of them onto another position on the screen of the computer, thus conserving their enclosing relationship (reported in category G5) b) by measuring both of them using spatial units (reported in category G7 and G8) c) by transforming automatically the minimum superset and making a parallel translation of the supplement of the non-convex polygon (reported in category G6). While the strategies included in the categories G5, G6 and G7 can be viewed as a sequence of two conservations the strategies included in category G8 can be viewed as a sequence of three. In this last category, students progressed to a calculation of the area of the non-convex polygon, expressing it in spatial units and then conserving its area by transforming it into an equivalent square using its area formulae.

Strategies in this approach can be characterized as advanced as their construction implies a solution plan as well as a view that integrates a number of different approaches to conservation. Moreover, by viewing the non-convex polygon as a subtraction of the two convex polygons described above, students were helped to overcome its irregularity.

The 'dynamic' approach. Students had the opportunity to approach the concept of conservation of area in an exploratory and dynamic way (reported in category G1) by studying a large number of different equivalent shapes automatically produced in this computer environment. Specific areas also were studied such as: classes of equivalent parallelograms and triangles with common bases and equal heights. By exploring the concept of conservation of area in the above classes of shapes students had the opportunity to form a more abstract view of this concept. It is not easy to provide students with this opportunity in a paper and pencil environment. Most students used the provided tools to transform automatically the non-convex polygon to a number of equivalent shapes. Since many students used this strategy as their initial solution strategy for the transformation task it probably indicates their preference for fast and easily delivered solutions (Kordaki & Potari, 2002) as well as a preference for solutions constructed by others and especially by the computer. Despite the fact that the possibility of conservation of area of the families of equivalent parallelograms was recognized by the students they could not justify their opinions, but relied on the computer's authority. In an attempt to construct their own argumentation three methods were developed by the students: a) the viewing of any new parallelogram produced from the previous one by cutting a triangle-area from its left part and pasting this triangle onto its right part, thus developing a sense of compensation for areas of the parallelograms they studied; b) automatically measuring the areas of different parallelograms included in each family and c) superimposing them using the appropriate tools of C.AR.ME. Regarding the conservation of area in the families of equivalent triangles, students started by not recognizing the possibility of equivalence in these shapes. To solve the conflict between their own opinions and the computers' authority, students developed the previously referred to methods (b) and (c) as well as they also studied the relation between the perimeter and the area of these triangles. In this way a deeper understanding of the concept of conservation of area regarding the above families of shapes was developed by some students. However, only a few students recognized the common properties of the shapes included in the above families; their common bases and their equal heights. Only these students used the area formulae to study the equivalence of the above shapes despite the fact that the students have been taught area formulae in school. Students also automatically transformed the areas under study to compare areas of the same form (reported categories G1, G3, G4) or to compare areas of the same form with the area-unit used (reported category G4). As certain automatically transformed shapes are automatically superimposed, students have the opportunity to make visual comparisons. The understanding of the concept of transitivity is also implied in the performance of these comparisons.

Students also used the tools for automatic transformations in combination with the other tools provided by this microworld and viewed the conservation of the non-convex polygon as a sequence of conservation (reported categories G3, G4, G6 and G9). These sequences of conservation are implied in the design of an appropriate solution plan, so they can be characterized as advanced approaches to conservation of area. In this way students seemed to attempt to synthesize different areas of knowledge; the computer and their own knowledge. The computer knowledge is reflected in the 'dynamic' approach while the students' knowledge is expressed by the 'intuitive', the 'spatial units' the 'enclosing in a superset' and the 'area formulae' approaches.

The most common path of students' strategies in performing the task of transformation consisted of the 'dynamic' and the 'intuitive' approaches to conservation of area. Despite this fact, most students also constructed advanced approaches to this concept. In this way they viewed the concept of conservation of area as a combination of different approaches to this concept as well as in integration with the concept of area measurement using spatial units and area formulae.

b) The role of tools

The use of conservation of area tools :

- *Tools that simulate the students' sensory-motor actions*. Students used these tools a) to express in a conscious way their previously referred to 'intuitive' approaches to the concept of conservation of area b) to experience in a spatial way the concepts of parallel translation of a figure, the reflection of a figure about an axis and the rotation of a figure through an angle about a point of rotation and c) to overcome the non-convex polygon's irregularity.
- *Tools for automatic transformations.* These tools were used by the students to construct the previously referred to 'dynamic' approach to the above concept. More specifically, these tools were used by the students a) to explore the concept of conservation of area in a plethora of shapes thus helping them to give a dynamic and more abstract character to this concept b) to investigate the above concept in families of equivalent parallelograms and of triangles with common bases and equal heights c) to overcome the non-convex polygon's irregularity d) to help students to make visual comparisons of the areas under study and d) to transform the given areas so they would have the same form as the area-unit used.

The use of area measurement tools:

- *Tools to construct a variety of spatial units and grids.* Students used these tools to conserve areas in a more sophisticated way by splitting them into equal parts and recomposing them to produce equal areas. By experiencing the above concept using different units or grids as representation systems, students were helped to form this concept in a more abstract way.
- *The tool for automatic area measurement*. Students used this tool to evaluate their conservation of area strategies. This tool was also used in relation to the tool for automatic measurement of length to overcome the confusion between the area and the perimeter of the shapes included in classes of equivalent parallelograms and of triangles in order to investigate the conservation of their areas.

7. CONCLUSIONS

This study suggests that the nature of the provided tools affected the students' constructions regarding the concept of conservation of area. Tools that simulated students' sensory-motor actions were used by the students in a conscious way to express actively their intuitive knowledge regarding this concept. In addition, tools that support automatic transformations of area gave students the opportunity to explore knowledge implied in these transformations. These tools were also used by the students in combination with the following: all tools of C.AR.ME., area formulae as well as the specific methods of 'enclosing the non convex polygon in a minimum convex super set' invented by the students to conserve its area. In general, students attempted to integrate the computer knowledge with their own knowledge.

Three main approaches to the concept of conservation of area of the non-convex polygon were developed by the pupils in this computer environment.

First, the ‘intuitive’ approach that indicates students’ intuitive knowledge of the concept of conservation of area in two ways. Firstly, by conserving the entire area of a shape changing only its position and secondly, by conserving the area of a shape after splitting and recomposing its parts. To perform this approach students used the tools that simulated sensory-motor actions in the context of C.AR.ME. in a conscious way at the same time exploring visually the basic area transformations of parallel translation and the rotation and symmetry of area which are implied in this process.

Second, the ‘enclosing in a minimum convex superset’ approach expresses the concept of conservation of the non-convex polygon as a sequence of conservation of two convex polygons. The first polygon is a minimum convex superset of the non-convex polygon and the second is the supplement of this polygon in relation to this minimum superset. Students constructed a number of approaches to conserve the area of the non-convex polygon by trying to combine this approach with those developed in this computer environment. As a result the concept of conservation of area was viewed as a sequence of different approaches to conservation.

Third, the ‘dynamic’ approach emphasizes the authority of the computer to automatically transform areas into equivalent geometrical shapes. By trying to justify this equivalence students looked more deeply into the concept of conservation of area. Moreover, students had the opportunity to explore this concept in classes of shapes of the same form such as in families of parallelograms and of triangles with common bases and equal heights. Even though the equivalence of these parallelograms was not questioned by most of the students, since they were relying on the computer’s authority, the equivalence of the triangles was not readily accepted. Students focused on the perimeter of these triangles and made conclusions regarding their areas. Students progressed to the understanding of the concept of conservation of area in both these classes of shapes by using the tools of C.AR.ME. More specifically, students used the automatic measurement of area and of length to study the area in relation to the perimeter of these shapes as well as the tools that simulated sensory-motor actions in the context of C.AR.ME. to superimpose the areas under study. By studying the concept of conservation of area in the innumerable variety of equivalent shapes produced automatically by the computer, students explored this concept in a dynamic way and formed a broader view.

In addition, most students viewed the concept of conservation of area as a combination of the above approaches or in integration with area measurement using spatial units or area formulae.

Finally, this study shows that the variety of the provided tools and the nature of tasks asking to be solved ‘in any possible way’ stimulated the students to express their own approaches to the concept of conservation of area as well as to construct a variety of different approaches to this concept.

I conclude with a brief note on the implications of this study for teaching the concept of conservation of area. The analysis of students’ work illuminated the possibilities of the tools of the microworld providing students with the opportunity of exploring this concept in specific classes of equivalent shapes, of expressing their own knowledge to this concept, of expressing different pieces of knowledge they possess constructing a variety of approaches, of building interrelationships between the various aspects of area and overcoming certain difficulties. It also shows the important role of the tasks that the teacher can plan to support students’ involvement in this environment. Hence it may help teachers to encourage their students: a) to explore the concept of conservation of area in a dynamic way regarding a plethora of equivalent shapes and to develop a broader view of this concept b) to explore the concept of conservation of area in a dynamic way regarding classes of equivalent parallelograms and triangles and c) to express their own knowledge about the concept of conservation of area. Further research is needed to investigate the role of the teacher and more specifically his or her interventions both at an individual level and in the classroom while exploiting this particular microworld.

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CAPTIONS

Figure 1. The general interface of C.AR.ME.

Figure 2. Examples of the use of tools that simulated the students' sensory-motor actions

Figure 3. Examples of the use of tools for automatic transformations

Figure 4. Students' transformation strategies based on the use of tools for automatic transformations

Figure 5. Students' comparison strategies based on the use of tools for automatic transformations

Figure 6. Examples of students' comparison strategies based on the use of tools for automatic transformations

Figure 7. Students' transformation strategies based on the use of simulations of students' sensory-motor actions

Figure 8. Students' comparison strategies based on the use of simulations of students' sensory-motor actions

Figure 9. Students' transformation strategies based on the use of simulations of students' sensory-motor actions in combination with the tools for automatic transformations.

Figure 10. Examples of students' transformation strategies based on the use of simulations of students' sensory-motor actions in combination with the tools for automatic transformations.

Figure 11. Students' comparison strategies based on the use of simulations of students' sensory-motor actions in combination with the tools for automatic transformations.

Table 1. Categories of students' strategies to face the task of transformation and of comparison in the context of C.AR.ME.

Table II. Students' strategies related to the concept of conservation of area across categories

File	**Draw**	**Edit**	**Automatic Measurements**	**Automatic Transformations**	**Measurement Tools**	**Help**
Open	Dot Square Grid	Select Part	Areas	Square	Square unit	
Open Last	Dot Triangle Grid	Select All	Angles	Rectangles	Rectangular unit	
Save Last	Draw Polygons	Cut	Segments	Families of Rectangles	Student unit	
Save As	Draw Segments	Paste		Parallelograms	Unit iteration	
Print	End Draw Polygons	Draw an angle of Rotation		Families of Parallelograms	Counting of units	
Exit	Clear	Rotate		Triangles	Square Grid	
		Draw an axis of Symmetry		Families of Triangles	Rectangular Grid	
		Symmetry about axis		Show numerical elements	Student Grid	
		Erasers				

Figure 1

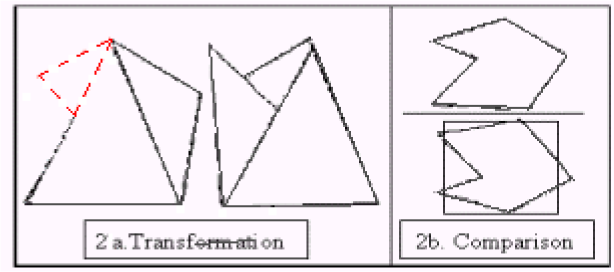

Figure 2

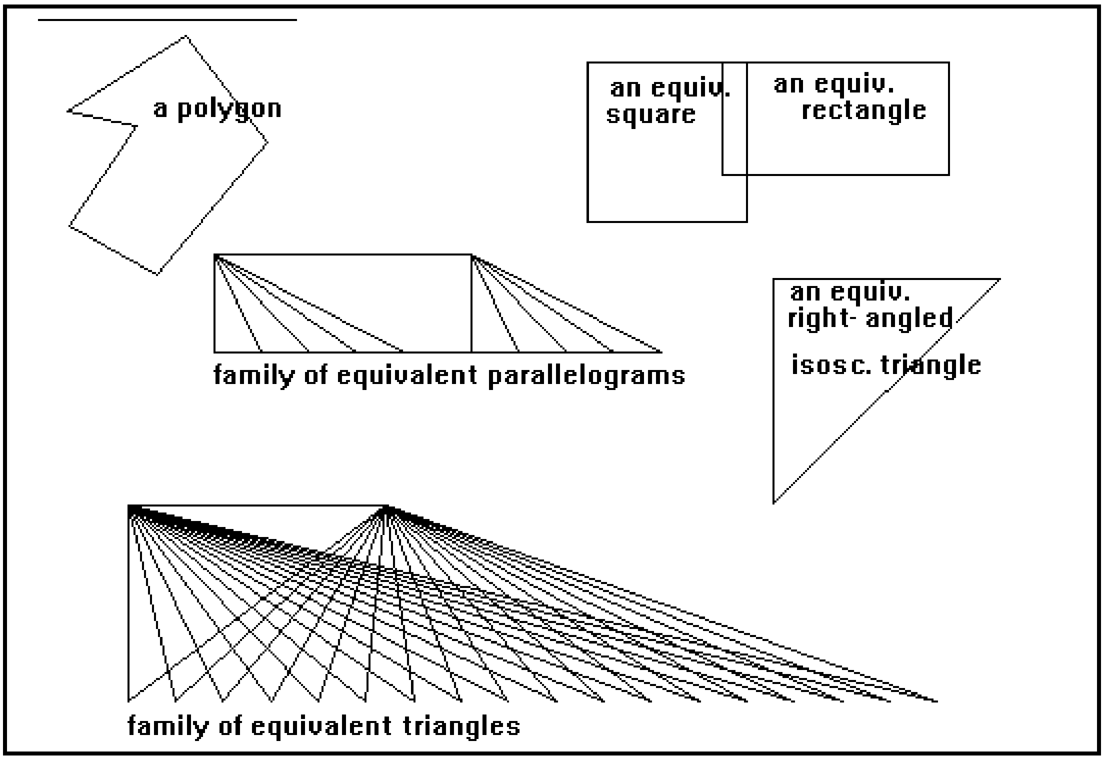
a polygon
an equiv. square
an equiv. rectangle
family of equivalent parallelograms
an equiv. right- angled isosc. triangle
family of equivalent triangles

Figure 3

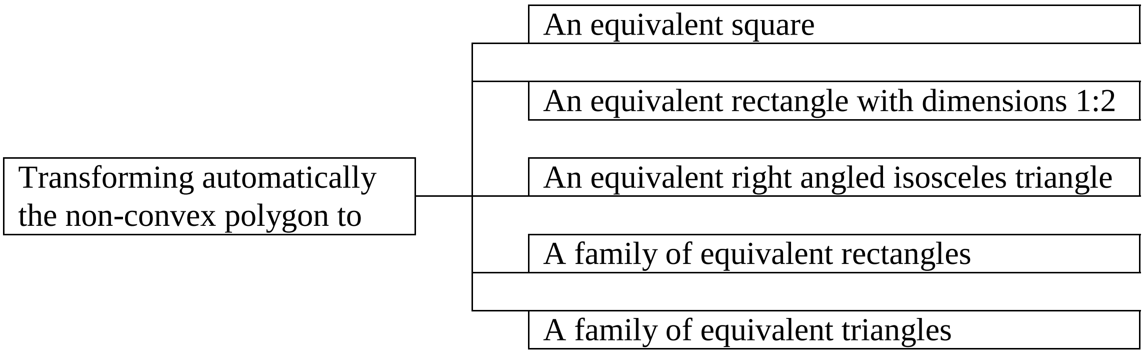
Transforming automatically the non-convex polygon to
An equivalent square
An equivalent rectangle with dimensions 1:2
An equivalent right angled isosceles triangle
A family of equivalent rectangles
A family of equivalent triangles

Figure 4

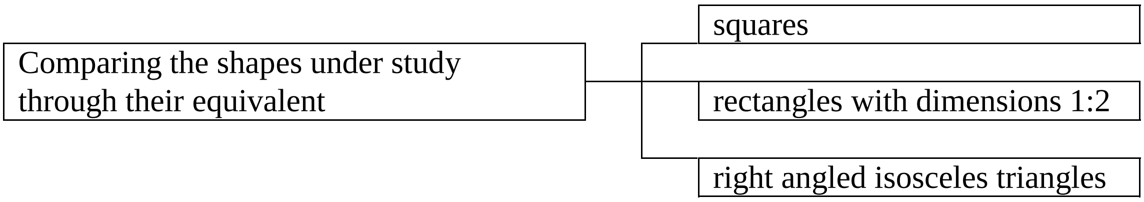
Comparing the shapes under study through their equivalent
squares
rectangles with dimensions 1:2
right angled isosceles triangles

Figure 5

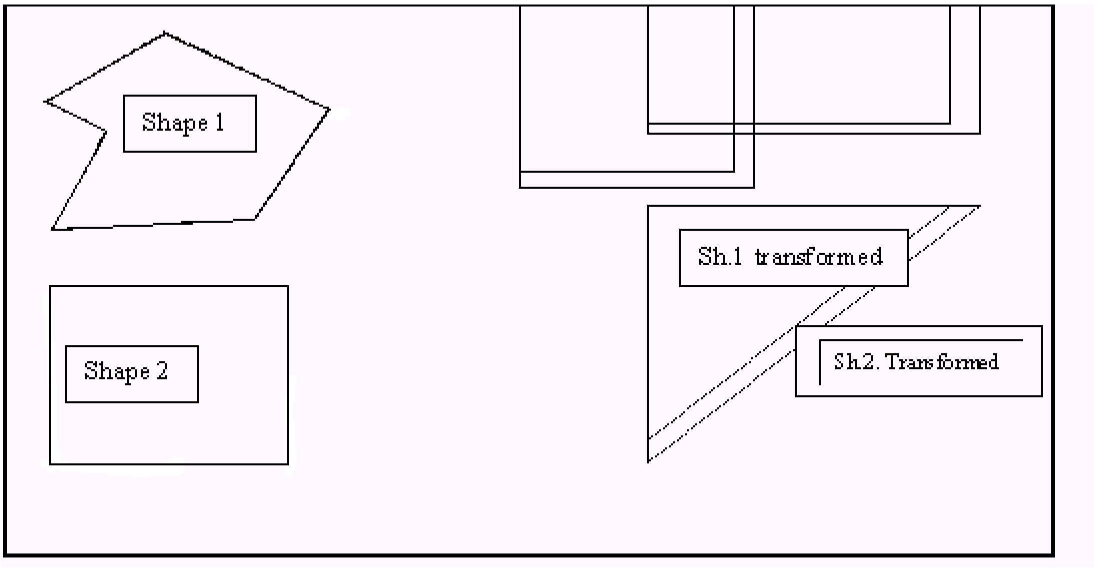

Figure 6

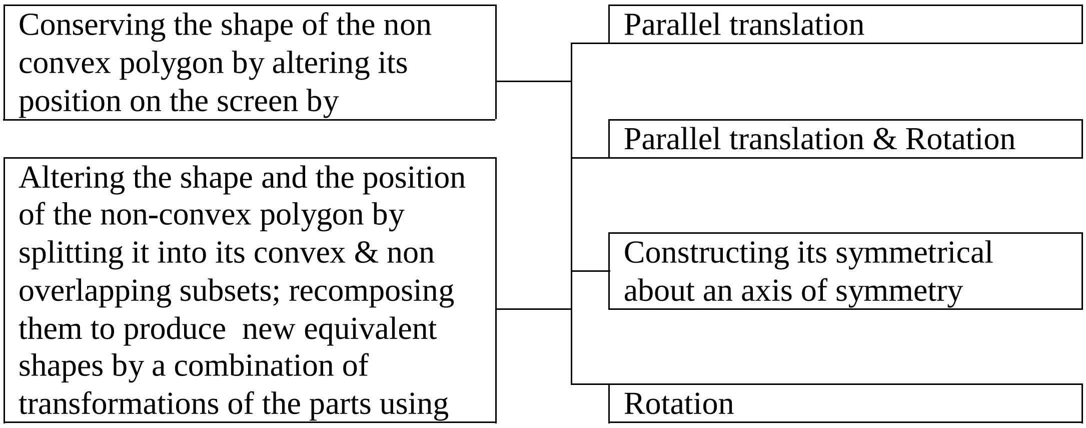
Conserving the shape of the non convex polygon by altering its position on the screen by
Altering the shape and the position of the non-convex polygon by splitting it into its convex & non overlapping subsets; recomposing them to produce new equivalent shapes by a combination of transformations of the parts using
Parallel translation
Parallel translation & Rotation
Constructing its symmetrical about an axis of symmetry
Rotation

Figure 7

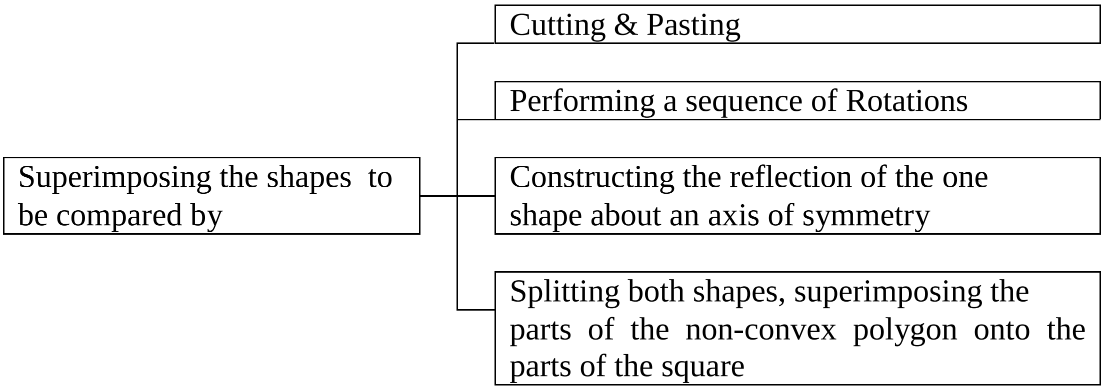
Superimposing the shapes to be compared by
Cutting & Pasting
Performing a sequence of Rotations
Constructing the reflection of the one shape about an axis of symmetry
Splitting both shapes, superimposing the parts of the non-convex polygon onto the parts of the square

Figure 8

Strategy A
• Splitting the non-convex polygon into convex parts; • Transforming automatically the parts to their equivalent squares or rectangles; • Recomposing the already transformed parts to form a new non-convex shape equivalent to the original
Strategy B
• Transforming automatically the non-convex polygon to its equivalent right angled isosceles triangle; • Splitting this into two equal triangles by drawing the altitude from the vertex of the right angle to the middle of the hypotenuse; • Recomposing these two triangles to form a new square equivalent to the original non-convex polygon
Strategy C
• Splitting the non-convex polygon into two convex parts; • Transforming automatically the parts to their equivalent squares; • Transforming the first of the squares to an equivalent rectangle with one side equal to the side of the other square; • Recomposing the second square and the rectangle to form a new rectangle, equivalent to the original non-convex polygon
Strategy D
• Transforming automatically the non-convex polygon to its equivalent rectangle; • Splitting this into two non equal rectangular parts; • Transforming automatically the first rectangle to an equivalent right angled triangle with one vertical side equal to one of the sides of the other rectangle; • Recomposing this triangle and the second rectangle to form a right angled trapezium equivalent to the original non-convex polygon

Figure 9

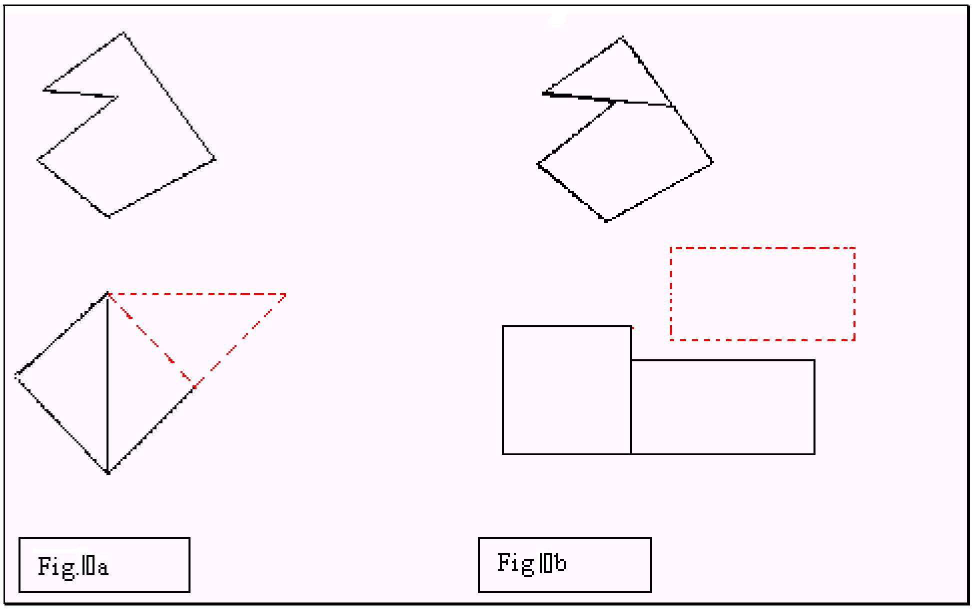

Figure 10

Strategy A
• Transforming automatically the non-convex polygon into its equivalent square; • Superposing this equivalent square onto the one under comparison; • Comparing the squares visually
Strategy B
• Transforming automatically the non-convex polygon to its equivalent square; • Moving this square side by side to the one under comparison so that their bases are aligned; • Comparing visually the sides of the squares to conclude about their areas
Strategy C
• Transforming automatically the non-convex polygon to its equivalent right angled isosceles triangle; • Splitting the square under comparison into two equal right angled isosceles triangles; • Superposing these triangles onto the equivalent triangle of the non-convex polygon; • Comparing the areas visually
Strategy D
• Splitting the non-convex polygon into two convex parts; • Transforming automatically the parts to their equivalent squares; • Superposing them to the square under comparison and; • Comparing the areas visually

Figure 11

- Measuring the non-convex polygon by using the square grid;
- Producing an equivalent square with the same number of area units;
- Transforming it automatically to an equivalent right – angled isosceles triangle and then to a rectangle

Figure 12

- Enclosing the non-convex polygon in its minimum convex superset;
- Parallel translation of this superset onto the screen of the computer;
- Copying the supplement of the non-convex polygon and subtracting it from the minimum superset

Figure 13

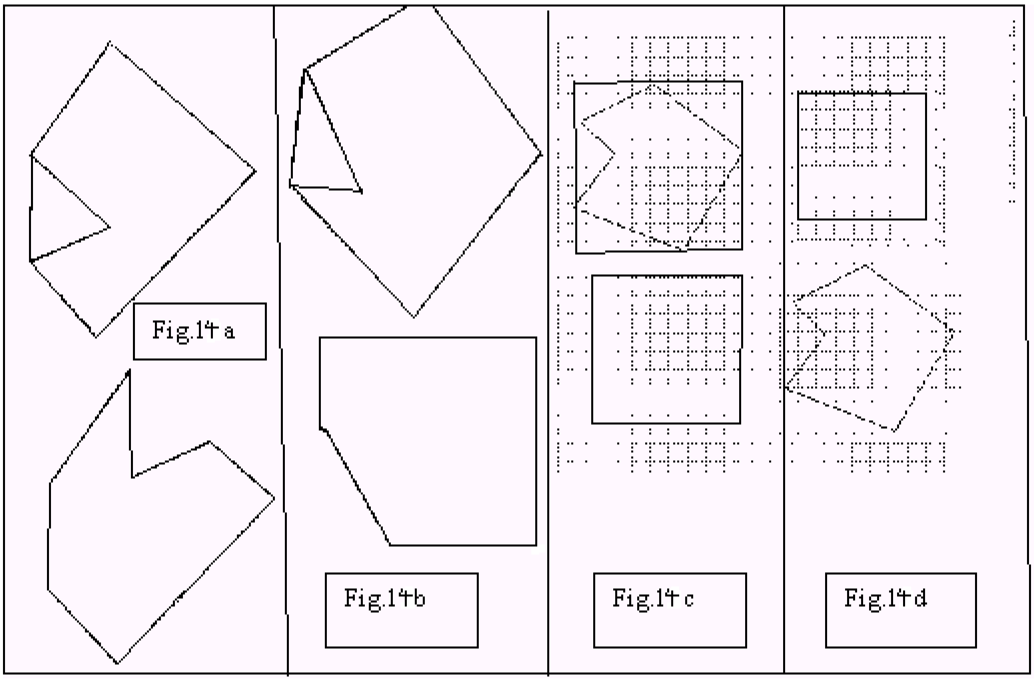

Figure 14

- Enclosing the non-convex polygon in its minimum convex superset;
- Copying the superset supplement relative to this non-convex polygon;
- Transforming automatically this superset to its equivalent square;
- Subtracting the supplement from this square

Figure 15

Strategy A
• Enclosing the non-convex polygon to a minimum rectangular superset; • Measuring the area of this superset and the area of the supplement of the non-convex polygon in relation to this superset using the square grid ; • Subtracting the results of this measurement in order to find the number of units needed for covering the non-convex polygon; • Measuring the area of the square using the square grid; • Comparing the original shapes by comparing the results of measurement (produced the previous way)
Strategy B
• Enclosing the non-convex polygon in a minimum square superset; • Measuring the area of this superset and the area of the supplement of the non-convex polygon in relation to this superset using the tool for automatic area measurement ; • Subtracting the results of this measurement in order to find the number of units needed for covering the non-convex polygon; • Transforming this area into square units by dividing it with the area of this unit. The square area unit was automatically measured by the students; • Measuring the area of the square using the square grid; • Comparing the original shapes by comparing the results of measurement (produced the previous way)

Figure 16

Transformation strategy
• Enclosing the non-convex polygon in a minimum rectangular/square superset; • Using the square grid to measure the area of this superset and the area of the supplement of the non-convex polygon in relation to this superset; • Subtracting the results of this measurement in order to find the number of units needed for covering the non-convex polygon; • Calculating the square root of this number of units; • Constructing a square with a side equal to the side of the unit used and this multiplied by the result of the previously calculated square root
Comparison strategy
• Enclosing the non-convex polygon in a minimum rectangular/square superset; • Measuring the area of this superset and the area of the supplement of the non-convex polygon in relation to this superset using a variety of area units; • Subtracting the results of this measurement in order to find the number of units needed for covering the non-convex polygon; • Calculating the square root of this number of units; • Constructing a square with a side equal to the side of the used unit multiplied by the result of the previously calculated square root; • Measuring the area of the original square by using the square unit and; • Comparing the original shapes by comparing the results of measurement (produced the previous way)

Figure 17

- Splitting the non-convex polygon into its convex & non overlapping sub- sets;
- Calculating its area by the automatic area measurement of the parts;
- Transforming the polygon to a square by using its area formulae and;
- an additional transformation automatically performed

Figure 18

Categories of students' transformation (C.T) and comparison (C.C) strategies in the context of C.AR.ME.			Number of students
Categories performed by using	**C.T**	**C.C**	
the 'eye'	C1	C1	1T, 1C
the perimeter of the shapes	C2	C2	4T, 2C
the tools for automatic transformations	**G1**	**G1**	28T, 10C
the simulation of students' sensory-motor actions	**G2**	**G2**	27T, 14C
the tools that support the operation of area measurement using spatial units	C6	C6	19T, 25C
the simulation of students' sensory-motor actions in combination with the tools for automatic transformations	**G3**	**G3**	8T, 16C
the tools that support the operation of area measurement using spatial units in combination with tools for automatic transformations	**G4**	C7	1T, 9 C
by enclosing the non-convex polygon in its minimum convex super set in combination with the simulations of students sensory-motor actions	**G5**	-	1T
by enclosing the non-convex polygon in its minimum convex super set in combination with the simulations of students sensory-motor actions and the tools for automatic transformations	**G6**	-	2T
by enclosing the non-convex polygon in a minimum convex super set in combination with the operation of area measurement using spatial units	-	**G7**	2C
by enclosing the non-convex polygon in a minimum convex super set in combination with the operation of area measurement using spatial units and the area formulae	**G8**	**G8**	2T, 2C
the area formulae	C8	-	1 T, 1C
the tool for automatic area measurement in combination with area formulae	C10	C10	1T, 1 C
the tool for automatic area measurement in combination with area formulae and the tools for automatic transformations	**G9**	-	5T
the tool for automatic area measurement	-	C3	23 C
the tool for automatic area measurement in combination with the simulation of students' sensory-motor actions	-	C4	4 C
the tool for automatic area measurement in combination with the tools for automatic transformations	-	C5	5 C
the area formulae in combination with the simulation of students' sensory-motor actions	-	C9	1 C
the operation of area measurement using spatial units in combination with area formulae	C11	-	2 T
	Sum = 14 cat.	Sum = 14 cat	

Table I

C : Task of comparison, T: Task of transformation

Students' strategies across categories

	The task of transformation							The task of comparison				
Students	**G1**	**G2**		**G3**	**G4**	**G5,G6 G8**	**G9**	**G1**	**G2b**	**G3**	**G7, G8**	**Sum str**
		G2a	**G2b**									
P1	**1**	**2,3,5**										4
P2	**6**	**1,4,5**	**7**					**6**	**3,4,5**			9
P3	**1**	**3**	**2**	**5**		**4 (G5)**				**1**		6
P4	**2**	**4,1,3,5**	**8**					**6,7,8**		**5**	**G8,G7**	12
P5	**1**	**2,3,4**		**5**		**6 (G6)**						6
P6	**2**	**6,3**							**5**	**6**		5
P7	**4**	**2,1,3**										4
P8	**1**	**4,2,3**							**2**	**5**		6
P9	**6**		**1**	**7**			**5**	**2**	**5,6**	**4**		8
P10	**1**	**2**	**4,6**	**7**				**1,3**				7
P11	**1**	**2,3**	**4,5,6**						**4**	**2**		8
P12	**1**	**2,3,4**										4
P13	**2**	**5,4**	**6**						**3**			5
P14	**1**			**8**	**6**	**7 (G8)**	**3,4**	**6**		**3**		8
P15	**3,4**		**5**	**6**					**5,3**	**4**		7
P16	**4**	**1,2,3**										4
P17	**2**		**3**				**1**		**6**	**4,7**		6
P18	**1**	**2,3**										3
P19	**1**	**5,6**	**4**						**4**		**5(G7)**	6
P20	**1**		**3**	**4,5,6**				**2,5**		**3,7**		9
P21	**3**	**2,4,5**							**4,5**			6
P22	**1**	**2,3**						**4**				4
P23	**1**						**2**	**4,5**		**3**		5
P24	**3**		**1**						**2**	**4**		4
P25			**1**							**4**	**2(G8)**	3
P26			**1**			**3 (G6)**			**1**	**4**		4
P27	**1**									**3**		2
P28	**1**	**2,3,4**						**2**	**3**			6
P29	**3**		**2**	**5**			**1**			**3**		5
P30	**1**	**2,6,7**	**5**			**8 (G8)**		**3**	**4**			8
Sum str	**28**	**48**	**19**	**10**	**1**	**5**	**6**	**15**	**19**	**18**	**4**	174
Sum Student	**28**	**19**	**16**	**8**	**1**	**5**	**5**	**10**	**14**	**16**	**3**	

Table II

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