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CHALLENGING STUDENTS TO VIEW THE CONCEPT OF AREA IN TRIANGLES IN A BROAD CONTEXT: EXPLOITING THE FEATURES OF CABRI-II

RUNNING HEAD: CABRI-TOOLS AND CONSERVATION OF AREA IN TRIANGLES

ABSTRACT

This study focuses on the constructions in terms of area and perimeter in equivalent triangles developed by students aged 12 to 15 years-old, using the tools provided by Cabri-Geometry II (Laborde, 1990). Twenty-five students participated in a learning experiment where they were asked to construct: a) pairs of equivalent triangles 'in as many ways as possible' and to study their area and their perimeter using any of the tools provided and b) 'any possible sequence of modifications of an original triangle into other equivalent ones'. As regards the concept of area and in contrast to a paper and pencil environment, Cabri provided students with different and potential opportunities in terms of: a) means of construction, b) control, c) variety of representations and d) linking representations, by exploiting its capability for continuous modifications. By exploiting these opportunities in the context of the given open tasks, students were helped by the tools provided to develop a broader view of the concept of area than the typical view they would construct in a typical paper and pencil environment.

KEY WORDS: Conservation of area in triangles, area and perimeter, Cabri-Geometry II, educational research, secondary education, problem solving.

INTRODUCTION

Area is an invariable attribute, a definite measurable size of the plane surfaces enclosed by figures which may be conserved while the shape of its figure is altered (Piaget, Inhelder & Szeminska, 1981; Douady and Perrin, 1986). Therefore, the construction of the identity of area implies a sufficient understanding of its conservation.

In traditional paper and pencil learning situations that involve concrete materials, students can master conservation of area through the *cut*, *move*, and *paste* actions in re-arranging the part of a shape to produce another shape with equal area (Piaget, et al., 1981). Understanding conservation of area from this perspective is a prerequisite for the understanding of area measurement using area-units and area formulae. Therefore, it is important to give students opportunities to study conservation of area, area measurement using area-units and area formulae in integration (Nunes, Light & Mason, 1993; Kidman & Cooper, 1997; Kordaki, 2003).

Understanding conservation of the global figure of a shape and consequently its area after its transformation through 'rigid movements' using translation, rotation and symmetry is also essential for the conceptualization of area. These transformations are called 'isometries' and are characterized mathematically as "preserving the distance".

In addition, understanding the invariance of an area is a process of giving meaning to its numerical, visual and symbolic representations (Nunes, et al., 1993; Kordaki & Potari, 2002). In particular, it is important to provide students with the opportunity to give meanings to different representations of areas in terms of: a) numerical representations produced by a measurement system (eg. an automatic area measurement tool), b) visual representations produced by using a representation system consisting of area-units (eg. a grid), and c) basic linear elements of a shape (eg. a base and the correspondent altitude) used in a symbolic representation system, such as area formulae.

Furthermore, the process of conserving an area and simultaneously studying it in relation to the perimeter of its figure is significant since students confuse these concepts and use them alternately (Piaget, et al., 1981; Hart, 1989; Kidman and Cooper, 1997).

Previous research on student understanding of conservation of area has been mainly based on the previously-mentioned Piagetian perspective. In these studies, the tools used have concentrated on the paper and scissors environment, with the students using their sensory-motor actions to cut up an area and then move and paste the parts to produce an equivalent area. The literature has also demonstrated that students frequently rely on their visual perception to make comparisons of areas, experiencing difficulties in understanding the possibility of conservation of area when represented in shapes of different forms (Carpenter, Coburn, Reys and Wilson, 1975; Hart, 1989; Tierney, Boys and Davis, 1986). Students also have difficulties with the form of the areas to be conserved; although they may understand the possibility of conservation of area in squares and in parallelograms, they face difficulties when attempting to conserve the area of irregular shapes and triangles (Maher and Beattys, 1986). However, the meanings that students can give to area are also closely related to the tools they use, the tasks they face and the shapes they study (Kordaki, 2003).

In particular, the role of tools provided by a computer microworld (the C.AR.ME. microworld [Conservation of Area and its Measurement], Kordaki and Potari, 1998), in combination with learning activities to be solved 'in as many ways as possible', were reported as being significant in the learning of area. These tools were used in a study that investigated student understanding of the concept of area and its invariance regarding non convex polygons as well as specific classes of both equivalent parallelograms and equivalent triangles with common bases and equal heights (Kordaki and Potari, 2002; Kordaki, 2003). Students presented difficulties in acknowledging the equality of the areas of those classes of triangles, since their perimeters were different, and in recognizing that the common properties of these triangles are their common bases and their equal altitudes. The role of Dynamic Geometry Systems (DGS) on students' constructions regarding congruent triangles has been also reported (Healy and Hoyles, 2001).

As for the concept of area - and in contrast to the paper and pencil environment or 'CA.R.ME.' micro-world - DGS provide students with different and potential opportunities in terms of: a) means of construction, by providing a rich set of tools, b) control, by using the 'drag mode' operation in combination with automatic measurement of area and length, c) the variety of representations, both numerical and visual, and d) linking representations, by exploiting its capability for continuous transformations.

Taking into account the previously mentioned opportunities provided by DGS as well as the experience from the aforementioned studies, we designed a learning experiment where students were encouraged to: a) study the concepts of area and perimeter in relation to each other, b) investigate these concepts within the context of scalene triangles, c) view the concept of area and its invariance in integration with: isometries, area measurement using area-units, area formulae, lines and polygons, d) express the different kinds of knowledge they possess through open learning activities, and e) exploit the advantages of a dynamic computer environment such as the well-known Cabri-Geometry II (Baturo and Nason, 1996; Moreira Baltar, 1997; Healy and Hoyles, 2001; Kordaki and Potari, 2002; Kordaki, 2003). While exploiting the features of Cabri (Laborde, 1990) within the context of the above learning setting, students were asked to perform, 'in as many ways as possible', specifically designed learning activities. Such an experiment has not yet been reported.

In this paper, we explore student strategies using Cabri tools for:

- conservation of area in triangles
- discrimination between concepts of area and perimeter in equivalent triangles

We also explore the role of Cabri tools in forming student strategies.

In the following section of this paper, we present our vision of Cabri as an appropriate environment for students' learning of area in terms of triangles, followed by a description of the context of the study and, subsequently, data analysis. Finally, the findings of the study are discussed and conclusions are presented.

CABRI AND THE CONCEPTS OF AREA AND PERIMETER IN TRIANGLES

As regards the concept of area, the paper-and-pencil based strategies emphasize splitting areas into parts (equal or not) and recomposing these parts to produce equal areas. Despite the fact that these strategies cannot easily be performed in the context of Cabri, this environment offers new possibilities for the conceptualization of area.

In particular, Cabri-Geometry II provides a rich set of tools regarding a variety of concepts concerning Euclidean Geometry. These tools can be exploited by the students to perform a number of different geometrical constructions leading to congruent and equivalent triangles. It is important to distinguish between those tools which lead to the construction of congruent triangles and those which lead to equivalent but not necessarily congruent triangles. The notion of congruence in triangles can be investigated by exploiting the direct availability of tools for: a) geometrical transformations such as symmetry, rotation, and translation, d) constructions of regular polygons and segments (to split these polygons into congruent triangles), e) constructions of parallel, perpendicular and arbitrary lines (to draw quadrilaterals and split them into congruent triangles as well as produce specific geometrical constructions of congruent triangles). In the context of Cabri, the notion of equivalence in triangles can be investigated by using: a) the grid provided b) specific constructions of triangles with equal bases and correspondent altitudes and c) the 'drag' mode operation in combination with the automatic area measurement tool. By using this combination of tools, students can control their constructions while exploring the notion of equivalence in triangles. It is worth noting that the 'drag mode' operation gives learners the possibility of forming dynamic views of the aforementioned concepts (Mariotti, 2001). More specifically, students have the possibility of handling, in a physical sense, the theoretical objects which appear as diagrams on the computer screen (Laborde and Laborde, 1995). In these Cabri-constructions, their geometrical properties are retained under dragging, while their visual output is different. The ‘drag mode’ can be used in two modes, as a ‘test’ mode and as a ‘search’ mode (Holzl, 2001, p.83).

The ‘drag mode’ operation can also be used in combination with the commands for automatic measurement of area and perimeter to help students to: a) primarily distinguish the concepts of area and perimeter in the context of congruent triangles, and b) understand that the equality of the area of equivalent but not congruent triangles does not imply the equality of their perimeter. The command for automatic measurement of area can be also used by the students as a verification tool for the equality of the areas of the triangles they construct.

Furthermore, by exploiting the variety of tools and operations provided, a number of continuous modifications of an original triangle into other triangles of equal area can be performed. In this way, classes of equivalent triangles including a number of different and linked representations in terms of the concept of area in triangles can be produced.

On the whole, the variety of geometrical tools and operations provided by Cabri for constructing and controlling a number of different linked representations of equivalent triangles can help students to: a) choose from among the tools those most appropriate for the expression of their knowledge, b) use a number of different tools to support the expression of different kinds of knowledge they possess, thereby developing a broader view of the concepts of area and perimeter in triangles than that they could construct in using paper, pencil and perhaps scissors and c) enhance their knowledge regarding area and perimeter by dynamically exploring the invariance of area in triangles.

THE CONTEXT OF THE STUDY

The focus and the methodology

This study focuses on students’ conceptions and their evolution concerning the concepts of area and perimeter in triangles while interacting with the Cabri-tools described in the previous section.

Student conceptions were expressed through their strategies in solving the problems of construction of pairs of equivalent triangles and of performing sequences of transformations (in terms of modifications) of an original triangle into other equivalent ones. This research is a qualitative study (Cohen and Manion, 1989) focusing on the variety of interactions realized by students working with Cabri tools as well as on the different ways that these students approached the conservation of area in triangles using these tools.

The learning experiment

The learning experiment took place in a typical, provincial, state secondary school in Patras, Greece. Three complete classes of students participated in this experiment, consisting of: eight 1st grade students (13-years-old), nine 2nd grade students (14-years-old) and eight 3rd grade students (15-years-old). These students were asked to perform two tasks using the Cabri II tools. The duration of tasks was commensurate with student needs, each student took about two hours to complete each task.

Before students commenced the main study, a familiarization phase, covering both basic operations of a computer and the Cabri II tools, was implemented, lasting two hours a week for each student over a period of approximately two months, the aim of which was simply to introduce students to the use of these tools and not proceed to the solving of the specific tasks. This phase was deemed necessary to support students in their effective use of the tools for the given tasks.

The specific operations were: turning the computer on/off, manipulating files (creating, opening, saving and printing a file) and using a word-editor. Students were then asked to try the specific Cabri tools that would enable them to experience only those concepts of Geometry which are taught in the Greek school curricula in the grades mentioned above, i.e. point, segment, lines, median, perpendicular bisector, circle, angle, triangle, regular polygons, symmetry, rotation and grid as well as calculations of area, angle and length. The tools presented were user-friendly and significant difficulties did not emerge.

Typical examples of tasks covered in the familiarization process included: ‘draw a line’, ‘draw a triangle’, ‘draw parallel lines’, ‘rotate a shape about a point’, ‘draw a shape and try the symmetry command’, ‘copy a shape’, ‘paste a shape’, ‘draw a shape and measure its area and perimeter’ etc. Moreover, students were introduced to the ‘drag’ mode operation, to the ‘automatic tabulation of numerical data’ as well as being instructed to use the ‘point on an object’ and ‘point on intersection’ tools. This was considered necessary because students usually use visual perception to add points to geometrical constructions when using paper and pencil.

During the familiarization phase, students were placed in groups of three and worked in a computer laboratory. These groups worked in rotation using three computers, whereas in the main study, while they also worked in rotation, due to technical limitations they used only one computer . The researcher (one of the authors) participated in the main study as an observer, with minimum intervention. This intervention is explained in a next section of this paper entitled ‘*The tasks’*.

Data resources in this study include the electronic files of students’ visual geometrical constructions, the video recordings of all interactions performed and the field notes of one of the researchers.

Students’ previous knowledge

All the students had been taught at primary school: a) how to calculate the area of a triangle by using the basic area formulae, A=1/2(B*H), (A= area of a triangle, B= a base of this triangle, H = the corresponding altitude to the base B), b) symmetry, c) regular polygons, d) the basic elements of a triangle such as sides, altitude, median, angle, perpendicular bisector, e) arbitrary, parallel and perpendicular lines, f) congruent triangles, g) measuring area using grids and h) perimeter. In school curricula, all these topics are encountered in isolation and conservation of area is not taught at all. Learning activities usually emphasize the calculation of areas using area formulae and rarely refer to area measurement using area-units. 3rd Grade students in this study had also been taught the criteria of congruence and similarity of triangles. It is worth mentioning that 1st grade school curricula emphasize the concept of a line (arbitrary, parallel and perpendicular) and the basic elements of a triangle mentioned above, while 2nd grade school curricula focus on area formulae. Students had also never been required to construct areas congruent and/or equivalent to an original area. In addition, no student had ever carried out activities where area was linked with different kinds of knowledge taught at school, such as regular polygons, lines etc.

Prior to undertaking the Cabri activities, all students were asked two questions by the researcher: 1. ‘Is it possible to have two or more triangles of different form with the same area?’ and 2. ‘What do you think about the area and perimeter of a shape: are these concepts the same or different?’. The point of this was to explore students’ previous knowledge of both congruence and equivalence in triangles as well as area and perimeter in order to investigate student progression through this learning experiment. From the data, it emerged that all students recognized the equivalence of area in congruent triangles but reported that they had never before considered that triangles of different form could have equal areas. Although all students had known how to verbally define area and perimeter since primary school, 3 out of 25 students expressed the belief that ‘area and perimeter are the same concepts’, the remainder reporting that they had never considered there was a difference.

The tasks

Two tasks were assigned, the first being: a) to ‘construct pairs of equivalent triangles, in as many ways as possible’ using Cabri tools (at this point, students were informed, by the researcher, that equivalent triangles are triangles with equal areas), b) to ‘justify your solution strategy’ and c) to explain ‘what you think about the relation of the area and perimeter of these triangles’. When the students seemed to be on the point of giving up, the researcher intervened, involving them in the task and encouraging them to continue by asking: ‘try another way of constructing another pair of triangles with equal areas. You can use other tools and the different kinds of knowledge you possess’. Students worked in groups of three to perform this task, so as to exploit the advantages of cooperation.

The learning aims of this task were to enable students to: i) advance smoothly from the notion of congruent triangles to the notion of equivalent triangles, beginning with the expression of their previous knowledge regarding congruent triangles and then enhancing their knowledge by exploring the equivalence in triangles using the ‘drag mode’ operation, ii) distinguish the concepts of area and perimeter in triangles by studying these concepts in relation to each other, and iii) link different kinds of knowledge about the concept of area through using the diversity of the tools provided, at the same time developing a broad view of this concept (Kordaki, 2003).

In the second task, students were asked to ‘construct a triangle and to perform any possible sequence of modifications to produce other triangles equivalent to the original’. More specifically, students were asked to: a) ‘construct an original triangle, then modify it into another equivalent triangle, using the Cabri tools’, b) ‘justify your solution strategy’ and c) ‘consider the produced triangle as the original triangle and repeat (a) and (b) as many times as you can, using different ways of modification’. The researcher intervened by encouraging the students to continue, as before. Students worked individually while performing the second task and it was decided to investigate how each individual student had perceived the learning experience of the first task after participating in the aforementioned group activity.

The additional learning aims of the second task were to enable each individual student to: i) construct individual approaches to the concepts of area and perimeter in triangles, ii) integrate different pieces of knowledge they possessed regarding the concept of area, by combining different strategies regarding the modification of a triangle into other triangles with equal area, iii) to develop a broad view of the concept of area and its invariance by constructing a class of triangles equivalent to an original triangle through a sequential process of modification.

Possible solution strategies: a-priori analysis

The potential strategies in constructing both; congruent and equivalent but not exclusively congruent triangles in the context of the provided tools are described in Figure 1.

(Insert Figure 1 about here)

Based on both the previously reported theoretical framework and the analysis of the strategies presented in Figure 1, we can state that:

Strategy S1, which is mainly based on students' visual perception (Strategy S1), is considered to be a primary approach to area (Piaget, et al., 1981), the other strategies reported in the above figures being considered more advanced.

Strategy S2 can be considered as a Cabri-transposition of the traditional geometrical processes used in a paper-pencil environment for constructing congruent triangles with ruler and compass. Cabri offers the possibility for exact construction by using the compass command if lengths are used, but this is not so easy if angles are considered. In the latter case, Cabri offers potential for approximate construction by measuring one or two angles of the initial triangle and using the rotation command. The construction is nevertheless much more complex than it would be in a paper and pencil environment. Of course, this approximate construction is also possible if the three lengths are considered by using the measurement tool.

Strategy S3 can be performed by the use of Cabri-commands for geometrical transformations. However, it can be mentioned that a translated triangle can also be obtained by the use of the editing commands (copy and paste). It is worth noting that applying the drag command to the original triangle leads to an infinity of translated congruent triangles. In the same way, using the rotate command with the original triangle produces infinity of congruent triangles obtained by rotation due to the characteristics of these commands, which are not mathematical ones.

Strategy G1 can be linked to S1 strategy. In S1, one can use exactly the same manipulation but with the purpose of constructing a congruent triangle by adjustment, while in G1 this strategy is used for accessing an equivalent non-congruent triangle. Consequently, G1 can spontaneously emerge from difficulties met with S1 if students use the measurement tool as a control tool and they can obtain the same measure even if the two triangles do not look congruent.

Strategy G2 can also be linked with G1. In particular, G2 can spontaneously emerge during the process of experimentation in the context of G1.

Strategy G4 is not at all easy to implement, especially if the vertices are not points of the grid or the area in integer or a very simple decimal or rational number, all the more so because the use of the measurement control will only allow very good approximate solutions.

Strategies linking different kinds of knowledge that students possess with conservation of area can be characterised as placing area more broadly in a context of geometrical concepts than while working within the traditional context of paper and pencil (Strategies S3, S4).

Strategies linking conservation of area with area measurement using area units and area formula can be characterised as expressions of area in different representation systems, thereby putting area in a broader context than in the traditional paper and pencil environment.

Strategies S3, S4, G2, G3, G4 and G5 can be viewed as being more advanced than S1, S2 and G1 as they necessitated the formation of a more complicated solution plan. In addition, strategies S3, S4, G1, G2, G3 and G4 are uncommon in the paper and pencil environment where the 'cut and paste' approach to conservation of area is mainly emphasized.

Finally, the expected modes that could be used by the students to justify their strategies in the context of Cabri could rely on control tools such as the measurement tools in combination to the 'drag mode' operation as well as on the geometrical properties used in the variety of strategies could be constructed. These justification modes are different from those modes that could be developed under the limitations of tools provided in the typical paper and pencil environment.

Data Analysis

Data were organized according to the two different tasks. All individual group multiple-solution strategies to the first task and individual student multiple-solution strategies to the second task were identified, reported and analyzed in terms of student conceptions of both area and perimeter, within the context of the equivalent triangles they had constructed. In the next stage of analysis, where the focus was on the students as a group, strategies were categorized in terms of the tools used. Next, student strategies in terms of both categories and grades were studied and, finally, the role of the tools provided in the construction of these strategies was investigated.

RESULTS

This section is in two parts, the results of both tasks being presented correspondingly. The results of both tasks are presented in terms of: a) categories of strategies, and b) strategies across categories and c) group strategies across grades. In the first task group experimentation in the area and perimeter of the constructed pairs of triangles is also presented.

Task 1: Constructions of pairs of equivalent triangles performed by the students 'in as many ways as possible' using Cabri -Geometry II tools

a) Categories of group strategies

Group solution strategies to this task were classified into ten categories, presented in Table 1. In this table, the specific groups and the number of students who performed strategies in each category are also presented. Capital letters A, B, C were used to represent the 1st, 2nd and 3rd grades, correspondingly. These letters are used in combination with the numbers 1, 2, 3, to represent the number of groups in each respective grade who performed the specific strategy.

Insert Table I about here

Students performed all types of strategies anticipated by the a-priori analysis. They also performed one more type of strategy (Strategy S5), to be presented in the following section. Here, all strategies are discussed in terms of how they were constructed by the students. It is worth mentioning that the participation process which took place in each group-work setting had a common characteristic: a participant proposed an idea for the solving process to the members of their group and this idea was tacitly accepted.

As students had the opportunity to select those tools most appropriate for the expression of their knowledge, all strategies they constructed were correct. Student difficulties that did occur are discussed with reference to the specific strategies constructed.

Categories of strategies leading to the construction of congruent triangles

S1: using the 'eye'. Students who performed this strategy constructed a scalene triangle on the computer screen using the 'triangle' tool. They then used the same tool to construct another triangle which appeared visually congruent to the original. To verify their solution strategy, these students measured the area of both triangles using the tool for automatic area measurement. Upon realizing that the areas were different, due to the area-unit implied in using this tool being one pixel, they were confused and expressed the opinion that '*it is impossible to construct equivalent triangles using our visual perception'*. To overcome this conflict, they adjusted the second triangle using the 'drag mode' operation. It is worth noting, that during the adjustment process, some students found a number of instances where this pair of triangles had the same area but not exactly the same figures. This fact encouraged these students to use the 'drag mode' in an exploratory mode in order to investigate if there are more triangles with same areas but different figures (Strategy G1).

S2: preserving lengths or lengths and angles of the original triangle. Here, students constructed an original triangle using the 'triangle' tool and then attempted to construct another triangle equivalent to the original by conserving the lengths of its sides. Despite the fact that this is a valid mathematical strategy, it was difficult to implement as it presupposes construction of an original triangle using the command triangle, its sides as segments, and then the congruent triangle through circle constructions using the compass command. Unfortunately, students tried to form the second triangle by chance, ignoring the previously mentioned geometrical constructions needed. In fact, students constructed a segment with equal length to one of the sides of the original triangle. Then, at the edges of this segment, they constructed two other segments with lengths equal to the corresponding two sides of the original triangle. To construct these three segments, students used the tool 'segment' the 'automatic measurement of length' and the drag command. By doing so, students simply produced a crooked line. Then, they tried to drag these segments to form a triangle but they failed to produce a triangle with equal sides to the original. To overcome this difficulty, students realized that one other possible solution would be to conserve the size of both the sides and the angles of the original triangle. At this point, students did not manage to form a new triangle either, as they had not conserved the order of both the sides and the angles of the original triangle.

S3: using Cabri-commands for geometrical transformations: Students who performed strategy S3a used the '*copy*' and '*paste*' tools to construct a triangle congruent to the original. Students did not use the 'translation' command as it had not been introduced to them during the familiarization phase because these students didn't know anything about vectors. So, a rather 'soft' construction was produced. Students, who performed strategy S3b, constructed an axis of symmetry and consecutively used the '*Reflection*' command. Students who performed strategy S3c assigned a center of symmetry and subsequently used the '*Symmetry*' tool. It is notable that the students of Grade A had been taught none of these isometries (except symmetry) at school but had been inspired to try them by the introduction of the transformation commands in Cabri during the familiarization phase.

S4: splitting polygons. Students who performed strategy S4a constructed an isosceles triangle as the original and split it into two equivalent triangles using the perpendicular bisector from the vertex of the equal sides. Although these students had previously been taught that each point of the perpendicular bisector of a segment is connected at its edges by a pair of segments of equal length, they were not sure of this and consequently measured the area of these two triangles for verification. While performing strategy S4b, students appeared to understand that a diagonal of a rectangle splits it into two right-angled congruent triangles. These students also confirmed the equivalence of the produced triangles using the tool for automatic area measurement. Regarding strategy S4c, students intuitively split a regular hexagon into six equivalent triangles using all its diagonals and then used the automatic area measurement tool to verify their equivalence.

S5: forming geometrical constructions producing pairs of congruent triangles. This strategy is not anticipated in the a-priori analysis. The students' strategy is described in Figure 2.

Insert Figure 2 about here

Here, students created a specific construction using parallel and perpendicular lines in a configuration where pairs of right-angled triangles were constructed, experimenting with the area of similar triangles ACZ and BCK by dragging point C onto segment ZK and finding that, when point C was located in the midpoint of the segment ZK, the area of these triangles was equal.

Insert Figure 3 about here

They then constructed a large number of pairs of right-angled triangles by dynamically rotating line e4 around point C and testing the equality of their areas (see Figure 3a). Students also used the 'drag mode' operation in combination with automatic area measurement, at the same time tabulating the numerical data produced, to verify this construction - using a large amount of visual and numerical data - as a method of constructing pairs of triangles with equal areas.

Categories of strategies leading to the construction of equivalent not exclusively congruent triangles

G1: using the 'drag' mode in combination with automatic area measurement. Students who performed this strategy experimented with two scalene triangles of different form by dynamically transforming the figure of the second triangle and simultaneously observing the variation of its area as demonstrated by the numerical results automatically produced and tabulated in a table (see Figure 3b). They found an instance where these triangles had equal areas and surprise was expressed that: '*two triangles of different form can have equal areas*'. It is worth noting that, during the experimentation process, some students observed that when a vertex of the second triangle moves in a parallel line from its opposite side the area of the produced triangles is conserved. In this way, this strategy was linked with the following strategy (Strategy G2).

G2: conserving the length of the base and its distance from its opposite vertex in a triangle. It is important to note that students who performed this strategy intuitively realized that the triangles produced had equal altitudes; it is likely they relied on their visual perception of the conservation of distance between parallel lines: they justified their strategy by saying that '*these triangles have common bases and seemed to have equal altitudes*'. Some students performed this strategy by studying a large number of equivalent triangles using the 'drag mode' while other students initially constructed a limited number of triangles. More specifically, these students constructed: two parallel lines e1 and e2, a segment BC on the line e1, a point A on the line e2, a triangle ABC, different points K, L on the line e2, the triangles KBC and LBC. Some of these students also went on to construct a large number of equivalent triangles using the 'drag mode' (see Figure 4a).

Insert Figure 4 about here

It is worth noting that all students who performed strategies in this category automatically measured the areas of the triangles produced while at the same time tabulating the numerical data, thereby verifying this strategy as a method of constructing a family of equivalent triangles.

G3: splitting a triangle into two equivalent triangles using a median. Students performed this strategy by splitting a scalene triangle ABC into two triangles using a median. Some of these students justified their constructions by expressing that '*these two triangles have equal bases and a common altitude, so according to area formulae these triangles have equal areas*'. In addition to this justification, they also verified their constructions using automatic area measurement. Although the remainder only expressed that '*these two triangles have equal bases*', they did justify their approach to producing equivalent triangles, by focusing on the results of the automatic area measurement tool used to measure them.

G4: measuring areas using area-units. By performing this strategy, students appeared to grasp that the area of a triangle can be conserved when the number of area-units needed to cover this triangle is conserved. Students correctly counted the whole units placed inside the triangle but faced difficulties in accurately estimating in terms of whole units, the parts of the units covering the remaining shape. As students tried to verify the equivalence of the constructed triangles, using the automatic area measurement tool, it was expressed that '*it is impossible to approximate the accuracy of the unit used by the computer'*. At this point, students made corrective adjustments to the second triangle using the 'drag mode' operation. It is worth noting that the verification of the equivalence of the constructed triangles using the automatic area measurement tool, where the area-unit used is one pixel, is more accurate than a verification based on the use of area-units included in the grid provided. Therefore, in performing this strategy, corrective adjustments to the second triangle using the 'drag mode' operation are necessary.

G5: using area formula. Three types of strategies included in this category (see Figure 5). Examples of student constructions are also presented in Figure 4b.

Insert figure 5 about here

Through performing Strategy G5a, students seemed to understand how to use area formulae in right-angled triangles and that the two perpendicular sides of such a triangle can be considered as its base and its respective altitude. They also seemed to understand that the area of a right-angled triangle can be conserved when the product of the lengths of its perpendicular sides is conserved, despite the fact that the figure of the triangle is altered. The same students progressed to the construction of strategy G5c, where they experienced the conservation of area in scalene triangles and demonstrated an understanding of the corresponding use of area formulae. Moreover, they appeared to understand the conservation of area of a scalene triangle as the conservation of the product of the lengths of its base and its respective altitude, even though the figure of this triangle can be transformed. Other students performed Strategy G5b and also correctly manipulated the area formula in a scalene triangle. Moreover, they realized that when its base remains unaltered and the respective altitude slides onto this base, the area of this triangle is conserved. Despite the fact that students who realized the strategies mentioned above used the area formula correctly in triangles, they needed verification of their strategies by automatically measuring all the equivalent triangles they constructed at the same time tabulating the numerical data produced. On the whole, however, when performing strategies in this category, students linked conservation of area with area formulae.

Area and perimeter. This section describes how students associated area and perimeter in this experiment. More specifically, students studied area in relation to perimeter while performing all previously mentioned group strategies, with students automatically attempting to measure the area and perimeter of the pairs of congruent and/or equivalent triangles they constructed. In all cases, students observed the numerical data produced by the automatic measurements mentioned above and noticed that '*the area of a triangle is different from its perimeter*'. All students also discovered that '*two triangles can have equal areas while having different perimeters*'. At this point, students were asked by the researcher to reflect on these different numerical data and to explain the difference between area and perimeter. The students answered: '*area refers to the space inside a triangle and perimeter to its boundary*', '*area is the interior of a triangle while perimeter is the curbed line around its figure*', '*area consists of all points included in a figure while perimeter consists of all points included in the curbed line surrounding this figure*', and '*area is the amount of space inside a triangle while perimeter is the sum of the length of its sides*'. On the whole, students seemed primarily to distinguish area and perimeter using the numerical outputs of the respective Cabri measurement tools. Students who experimented with a large number of equivalent triangles using the 'drag mode' (strategies fall in category G1) discovered that the equilateral triangle has the minimum perimeter.

b) Group strategies across categories

Group strategies across categories are presented in Table II. The number in each cell indicates the order of the performance of the specific strategy corresponding to each group of students. For example, Group A3 performed four strategies in total, the first falling into category G1, the second into category G2, the third into category S3 and the fourth into category S4. The last column of this table shows the progression of strategies performed by each group, with some groups performing more than one strategy in category S3, as this contains three different types of strategy. In this column, strategies leading to equivalent not exclusively congruent triangles are presented in bold face.

Insert Table II about here

As shown in Table II, all groups performed strategies leading to the construction of both congruent and equivalent but not exclusively congruent triangles. The most common group strategies for the construction of congruent triangles were based on the use of isometries (21 strategies in total, performed by 7 groups) while those for equivalent but not exclusively congruent triangles were constructed through experimenting with the 'drag mode' operation (23 strategies, performed by 8 groups). Despite the fact that there is no clear order between the strategies, one can notice that S1 and S3 (leading to congruent triangles) are the two main first strategies (7 cases). G5 and the non-anticipated S5 were the less used strategies. It was probably due to the fact that they demand the construction of a complicate solution plan. At some exceptions, strategies seem to be used only once.

The minimum number of strategies performed by each group was two, with the maximum being eleven and the average number of strategies performed being five strategies per group. The diversity of strategies developed by the groups does not seem to be dependent on the grade they are in.

Motivation. Although a number of groups (four groups) initially constructed strategies mainly based on visual perception (category S1), these groups were motivated to construct more advanced strategies due to the nature of the given task, which was to be solved 'in as many ways as possible', as well as by the rich variety of tools provide by Cabri II. Moreover, the existence of the relevant tools provided in the Cabri interface encouraged students to express their previous knowledge of isometries and polygons. Students were also curious to experiment with the area of triangles of different form to find out if it was possible to conserve their area; in this, they were motivated by using the 'drag mode' operation. In our view, this fact indicates that the 'drag mode' operation is a significant feature that enables students to experiment and to move from the notion of congruent triangles to the notion of equivalent triangles.

Task 2: Sequences of modifications of an original triangle performed by students using Cabri - tools

a) Categories of student strategies

Students successfully integrated the strategies mentioned in the first task (see Table I) into a sequential modification process where each student constructed an original triangle on the computer screen by using the 'triangle' tool and then performed one of the modification strategies mentioned above. Next, the equivalent triangle produced was taken to be the original and the student performed another modification on it. Modifications were repeated until the students had exhausted their modification strategies. Strategy that fall in category S5 was not performed by any student in this task. One explanation for this could be that students were probably unable to integrate this strategy into the sequential modification process. Moreover, students performed all strategies included in category S3 plus one strategy for the transformation of a triangle through rotation (Strategy S3d). At this point, it is worth noting that all categories of the performed strategies, excepting those that fall into categories S4 and G3 (splitting polygons and splitting a triangle using a median correspondingly), are based on modifications of an original triangle. Consequently, it is important to demonstrate how students constructed triangles equivalent to the initial one using these ‘splitting’ strategies. The students’ constructions are presented in Figure 6.

Insert Figure 6 about here

b) <u>Student strategies across categories</u>

Table III demonstrates the sequence of modifications of the area of an original triangle performed by each student who participated in this experiment. This Table is organized in the same way as Table II.

Insert Table III about here

As is shown in Table III, all students performed strategies leading to both congruent (75 strategies in total) and equivalent but not exclusively congruent triangles (48 strategies in total). The most common strategies that students performed were: a) Isometries: Almost all students (24 of 25 students) performed more than one different strategy (S3a, S3b, S3c) in this category (56 strategies in total), while the students’ first, second and third transformation strategies mainly fall into this category, b) Almost all students (24 of 25 students) also performed strategies which lead to the construction of non-congruent but equivalent triangles through dynamic transformations based on the use of the ‘drag mode’ operation in combination with the display of area (G1 and G2; in total 33 strategies), c) A considerable number of students (16 out of 25 students) performed strategies leading to equivalent but non-congruent triangles, where area formulae and area measurement using area-units were combined with the ‘drag mode’ operation (G3, G5 and G4 correspondingly).

It is worth mentioning that the initial strategies of most students (20 students) were based on their visual perception and previous knowledge (S1, S3, S4, G4). In addition, a considerable number of students (16 out of 25 students) started to construct strategies leading to congruent triangles but later progressed to the construction of strategies leading to equivalent but non-congruent triangles. That student’s reverse to the construction of congruent triangles was not due to interventions of the researcher, as none were made, but can be explained by the wealth of strategies they were able to devise.

Moreover, most students (18 in all) did not repeat the performance of strategies based on the use of area-units in the second task. In our view, this is an indication of student difficulty in integrating this measurement approach into the sequential transformation process, as this presents complications in accurately estimating an area in terms of area-units in this computer environment.

Hence, the number of students who performed strategies based on their visual perception decreased during the second task (6 students did not repeat this strategy). In our view, this indicates that these students have progressed in their ability to construct more advanced transformation strategies.

As to the number of strategies performed in both tasks, 9 students performed more strategies in the second task, 13 students performed more strategies in the first task while the remaining 3 students performed the same number of strategies in both tasks. Hence, the minimum, maximum and the average number of strategies were very similar in both tasks. In our view, this can be explained by the fact that both tasks drew upon the same store of strategies. Finally, the diversity of strategies again did not seem dependent on the grade the students were in at school.

DISCUSSION

A first glance at these results shows that the students have performed a variety of constructions of equivalent triangles using the Cabri-Geometry II tools. A total of one hundred and thirty seven constructions were performed by the 25 students carrying out the first task in this experiment, these being classified into ten categories. Students were prompted to perform this wealth of constructions by exploiting the variety of the tools provided and by being asked to approach the task ‘in as many ways as possible’. Most of these constructions (nine of those categories mentioned above) were used by students in combination to form classes of triangles equivalent to an original triangle through a sequential transformation process. During this process, the students performed one hundred and twenty three specific transformations in total. All students were also actively involved in both tasks and each performed at least two correct strategies for each task.

Concentrating more deeply on the results of this study, while at the same time bearing in mind the learning aims of both tasks, we discuss in the following section the constructed strategies in terms of: i) student learning and ii) the role played by the tools provided in student constructions.

i) Student learning expressed through their strategies

Students seemed to view the concepts of area and perimeter regarding triangles in a broader context than when in a typical paper and pencil environment by: a) advancing from the notion of congruence to the notion of equivalence in triangles, b) giving a variety of meanings to the area of a triangle using different measurement representation systems, c) linking and integrating different kinds of knowledge with the concept of area in triangles, d) moving from primary strategies to more advanced ones, e) constructing classes of equivalent triangles, f) primarily discriminating the concepts of area and perimeter in triangles and g) devising methods uncommon to paper and pencil practice for the construction of pairs of equivalent triangles. These are detailed in the following section.

Advancing from the notion of congruence to the notion of equivalence in triangles. All students were attracted by the ‘drag mode’ operation and experimented with it in combination with the display of area. By using these features, all students constructed an abundance of equivalent but non-congruent triangles, at the same time recognizing that the area of a triangle can be conserved despite the fact that its figure can be altered. Some students also progressed in their understanding of the concept of area and its invariance by using the ‘drag mode’ operation dynamically to construct families of equivalent triangles with common bases and equal altitudes. Other students enhanced their knowledge of area formulae in triangles by observing visual dynamic representations in a large number of triangles with common bases and equal altitudes; these students appeared to link conservation of area with area formulae. In addition, some students proceeded to construct equivalent triangles by using area measurement in terms of area-units with the use of grid in combination with the use of the 'drag mode' operation. In this way, conservation of area was connected with area measurement using area-units. Equivalent triangles were also constructed by splitting a scalene triangle using a median. By dynamically altering the figure of a triangle using the 'drag mode' operation, provided as a basic option by Cabri, and automatically measuring its area, students had the opportunity to investigate the invariance of area in a large number of equivalent triangles of different figures and to move from the notion of congruence to the notion of equivalence in triangles. This is impossible to realize in a paper and pencil environment.

Giving a variety of meanings to the area of a triangle using different measurement representation systems. Students had the opportunity to give a variety of meanings to the concept of area in triangles by studying its conservation in the context of different measurement representation systems, such as: a) the automatic area measurement system, where areas are represented as numbers, b) the area-unit measurement system, where visual representations of area are produced using the provided grid and c) the symbolic representation system of area-formulae, where areas are represented in terms of their basic linear elements (bases and the correspondent altitudes). It is worth noting that all students used the system (a), six students used the system (b), ten students used the (c), and four students used (a), (b) and (c) systems.

Linking and integrating different kinds of knowledge with the concept of area in triangles. During the first task, all students linked the different kinds of knowledge they possessed with the concept of area in triangles. More specifically, students linked the concept of area with: a) the perimeter (in all strategies that students constructed), b) area measurement using area-units, c) area measurement using area formulae, d) isometries, e) basic elements of a triangle such as altitude, median and the perpendicular bisector of the base of an isosceles triangle, f) geometrical constructions using parallel, perpendicular and arbitrary lines, g) regular polygons, h) typical quadrilaterals, such as rectangles and squares, and i) fractions. In the second task, all students integrated the different kinds of knowledge mentioned above (except f) with the concept of area.

Students demonstrated their previous knowledge in constructing congruent triangles as well as non-congruent but equivalent triangles. Students were encouraged to express their previous knowledge by the presence of the associated tools on the Cabri interface, at the same time bearing in mind that both tasks were to be solved “in as many ways as possible”.

Progressing from primary strategies to more advanced ones. Students realized that it is impossible to construct pairs of congruent triangles using their visual perception by measuring these automatically. They were encouraged to progress from these primary strategies by the ‘drag mode’ operation and by the variety of tools in the Cabri interface.

Constructing classes of equivalent triangles. Each student exploited the capability of Cabri for continuous modifications, the diversity of the tools provided as well as their experience from the first task and combined different strategies (minimum 2 strategies, maximum 6 and mean 4.9 strategies) to construct a class of triangles equivalent to an original one during the second task. Students integrated different kinds of knowledge they possessed (mentioned above) and constructed classes including a large number of congruent and equivalent but non congruent triangles. Most strategies constructed during the first task were used by the students in their attempts to form the classes mentioned above, with the exception of the specific construction producing pairs of right-angled equivalent triangles, which was not repeated. In our view, this indicates that, as this strategy did not involve a transformation, students were unable to integrate it into the sequential transformation process demanded by the second task.

Primarily discriminating between the concepts of area and perimeter in triangles. All students studied area in relation to perimeter by using the ‘drag mode’ operation in combination with display of area and display of perimeter. By using this combination of tools, all students provided with a considerable amount of empirical data that helped them to primarily discriminate between the notion of area and perimeter in congruent and equivalent triangles. Moreover, all students recognized that equality of the area of non-congruent but equivalent triangles does not mean equality of their perimeters. Students also helped to primarily discriminate between the concepts of area and perimeter by studying them in relation to each other in a variety of triangles of different figures, such as scalene, isosceles, equilateral, right-angled triangles and classes of equivalent scalene triangles with common bases and equal heights.

Devising methods uncommon to paper and pencil practice for the construction of pairs of equivalent triangles. In order to construct congruent triangles, students devised methods uncommon to paper and pencil practice such as: a) splitting isosceles triangles using the perpendicular bisectors of their bases, quadrilaterals and regular hexagons using their diagonals, b) forming a specific geometrical construction producing pairs of congruent triangles using two parallel lines with a perpendicular segment and a secant line from the midpoint of this segment to the previously mentioned parallel lines. By rotating the secant line around this midpoint, students had the opportunity to investigate the conservation of area in different classes of pairs of congruent triangles. Students also devised methods uncommon to paper and pencil practice in order to construct pairs of equivalent but not exclusively congruent triangles by splitting scalene triangles using a median.

ii) The role of tools in assisting student constructions

The availability of a diversity of tools. The presence on the Cabri interface of a variety of tools enabled the students to: a) express the different kind of knowledge they possessed and to construct a variety of geometrical constructions leading to congruent and equivalent not exclusively congruent triangles, with each student constructing at least two correct solution strategies per task and selecting from the tools provided those most appropriate for their knowledge, b) link and integrate different kinds of knowledge with the concept of area in triangles, at the same time putting the concept of area in triangles into a broad context of geometrical concepts, c) move from primary strategies to more advanced ones and d) devise methods uncommon to paper and pencil practice for the construction of pairs of equivalent triangles.

Control tools: a) the 'drag mode' operation, b) the automatic area measurement, c) the automatic measurement of perimeter and d) the automatic tabulation of numerical data. By combining the (a), (b) and (d) tools, students were challenged to observe a large number of non-congruent but equivalent triangles, to verify the conservation of their area as well as to move from the notion of congruence to the notion of equivalence in triangles. In addition, these features were used as scaffolding elements to improve the construction of equivalent triangles based totally or partly on students' visual perception. Furthermore, the combination of (b) and (c) tools -used in all student strategies- helped them to discriminate primarily between the concepts of area and perimeter.

Students used the combination of tools (a) and (b) in three modes: firstly, in an *exploratory mode* where they investigated the existence of scalene triangles which conserve their area as well as the possibility of conservation of area in classes of triangles with a common base and opposite vertex scrolling on a line parallel to its base; secondly, in a *verification mode* to verify the validity of their constructions of equivalent triangles and, thirdly, in an *adjustment mode* to refine their constructions of equivalent triangles based totally or partly on their visual estimation of areas. This adjustment process also helped some students to progress from the notion of congruence to the notion of equivalence in triangles. On the whole, students took to the 'drag mode' operation and tried to use this feature in their constructions.

The automatic area measurement tool was also used in every strategy performed by the students, as a control tool in order to verify their attempts at constructing equivalent triangles in both tasks.

The possibility of constructing different representations of area. The variety of tools provided helped students to construct different representations of area in triangles using the diversity of geometrical concepts described in the previously mentioned section. Students expressed these different pieces of knowledge qualitatively by giving their constructions visual and spatial meanings. Students also constructed new means of control of their strategies by verifying them using the numerical results produced by using the tool for automatic area measurement. This enables us to state that students approached area and its invariance in triangles from a ‘semi-qualitative’ perspective. Students also discriminated between the concepts of area and perimeter by using the tools for automatic area measurement and length in relation to the ‘drag mode’ operation. To this end, students approached the relation of these concepts quantitatively.

The possibility of there being continuous modifications. Students exploited the possibility of performing continuous modifications of an original triangle in combination with the diversity of tools provided to integrate different kinds of knowledge they possessed into the formation of classes of equivalent triangles.

It is worth noting that the paper and pencil based strategies regarding with area measurement using area units as well as the construction of congruent triangles by preserving lengths or lengths and angles of an original triangle were not easily performed by the students in the context of Cabri. In addition, splitting areas in parts and recomposing these parts to produce equal areas was not performed during this experiment as Cabri does not offer this possibility. These strategies are highly consistent with the Piagetian perspective and with learning situations that perhaps involve the use of concrete materials. However, Cabri offered to the students a different view, potentially at least: in the light of Cabri-tools other means of thinking about conservation of area were taken into account by the students while constructing their strategies.

CONCLUSIONS

Our exploratory study points to a direction of research that, in our opinion, is very promising. The basic aim concerned the integration of a learning concept into a broad context consisting of a variety of geometrical concepts. Our main hypothesis claimed the possibility of the enhancement of student knowledge of a learning concept by using a rich set of relevant tools provided by Cabri while performing appropriately-designed learning activities. An essential characteristic of these activities was that they be solved ‘in as many ways as possible’.

This study supports our hypothesis by demonstrating that the selected set of Cabri-tools and its dynamic character, in combination with the openness of the given tasks, inspired the students to view the concept of area in triangles in a broad context. More specifically, this study shows that students did use the selected set of tools provided by Cabri 'in as many ways as possible' and actively performed an abundance of strategies to construct equivalent triangles. They then used most of these strategies in a variety of sequential transformation processes in order to form classes of triangles equivalent to an original triangle. By exploiting the variety of the tools provided and the dynamic character of Cabri in the context of the given open tasks, students: a) progressed from the notion of congruence to the notion of equivalence in triangles, b) assigned different meanings to the concept of area by studying it in different measurement representation systems, c) linked and integrated different kinds of knowledge with the concept of area in triangles, d) moved from primary strategies to more advanced ones, e) constructed classes of equivalent triangles, f) discriminated primarily between the concepts of area and perimeter in triangles and g) devised methods uncommon to paper and pencil practice for the construction of equivalent triangles.

The nature of the selected Cabri-tools affected student constructions by providing them with new means of: a) *constructions*: students exploited the presence of a diversity of tools in the Cabri interface and expressed different pieces of knowledge they possessed, namely: perimeter, area measurement using area-units, area formulae, isometries, polygons, basic elements of a triangle, different types of lines as well as arbitrary, parallel and perpendicular lines. By expressing these different pieces of knowledge, students had the opportunity to enhance their understanding of concepts of area and perimeter in triangles. In addition, students used this knowledge to devise methods of construction of equivalent triangles uncommon to typical paper and pencil practice. b) *control*: by using the 'drag mode' provided by Cabri in combination with display of area and display of perimeter, students were helped to progress from the notion of congruence to the notion of equivalence and could primarily discriminate between the concepts of area and perimeter. Students also developed new modes of justification for their strategies, based on the use of these tools. c) *representations of area:* numerical and visual, and d) *linking representations of area:* by exploiting the Cabri capability for continuous transformations.

Finally, the availability of a variety of tools regarding different aspects of Euclidean Geometry in the Cabri environment, its dynamic character and the automatic measurement tools provided as well as its capability for continuous modifications makes it a powerful environment for both the teaching and learning of the concept of area in triangles as well as the study of this concept in relation to the perimeter of these shapes. On the whole, this research shows that regarding areas in the light of Cabri-tools suggests new ways of thinking about the development of the concept of area with respect to those evoked at the beginning of this article.

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CAPTIONS

Table 1. Categories of student strategies for the construction of equivalent triangles ‘in as many ways as possible’ using Cabri II tools.

Table II. Group strategies across categories for the construction of equivalent triangles ‘in as many ways as possible’ using Cabri II tools.

Table III. Student strategies across categories regarding the construction of any possible sequence of transformations of an original triangle using Cabri II tools.

Figure 1. Categories of possible strategies leading to the construction of congruent and equivalent not exclusively congruent triangles in the context of Cabri-Geometry II

Figure 2. Student strategies performed by forming geometrical constructions producing pairs of congruent triangles

Figure 3. Student constructions regarding strategies fall in categories S5 and G1

Figure 4. Student constructions regarding strategies fall in categories G2 and G5

Figure 5. Student strategies performed by using area formula

Figure 6. Student strategies for the transformation of an original triangle to other equivalent ones by splitting polygons

Categories of group strategies regarding the construction of equivalent triangles in the context of Cabri-Geometry II	**Number of groups**	**Number of students**
Strategies leading to congruent triangles		
Categories performed by		
S1: using the 'eye'	1A, 1B, 2C	2A, 3B, 5C
S2: preserving lengths or lengths and angles of the original triangle	1B	3B
S3: using Cabri-commands for geometrical transformations	2A, 3B, 2C	6A, 9B, 6C
S4: splitting polygons	3A, 1C	8A, 3C
S5: forming geometrical constructions producing pairs of congruent triangles	1A	3A
Strategies leading to equivalent not exclusively congruent triangles		
Categories performed by		
G1: using the 'drag' mode in combination with automatic area measurement	2A, 3B, 3C	6A, 9B, 8C
G2: conserving the length of the base and its distance from its opposite vertex in a triangle	3A, 1B, 1C	8A, 3B, 3C
G3: splitting a triangle using a median	1A, 2B, 1C	3A, 6B, 3C
G4: measuring areas using area-units	1A, 2B, 1C	3A, 6B, 3C
G5: using area formulae	2B	6B

Table I

The task of construction of pairs of equivalent triangles 'in as many ways as possible' using Cabri II tools													
Grades	**Groups**	**Categories of strategies leading to..**										**Total Strat.**	**Path of categories/group**
		Congruent triangles					**Equivalent not exclusively congruent triangles**						
		S1	**S2**	**S3**	**S4**	**S5**	**G1**	**G2**	**G3**	**G4**	**G5**		
1st	**A1**			1,8,9	6,7	10	2	4,5	11	3		11	S3,**G1,G4,G2**, G2, S4, S4, S3, S3, S5,**G3**
	A2	1			2			3				3	S1,S4,**G2**
	A3			3	4		1	2				4	**G1,G2**,S3,S4
2nd	**B1**			1	6		2	3	7	4	5	7	S3,**G1,G2,G4,G5**,S4,**G3**
	B2	1		3			2					3	S1,**G1,**S3
	B3		2,3	6,7			4		8	5	1	8	**G5**,S2,**G1,G4,**S3, S3,**G3**
3rd	**C1**	1		2			3			4		4	S1,S3,**G1,G4**
	C2			1,3,4			5	6	2			6	S3,**G3,** S3,S3,**G1,G2**
	C3	1					2					2	S1,**G1**
Total students		10	3	21	11	3	23	14	12	12	6		

Table II

The task of transformation of an original triangle 'in any possible sequence of transformations' using Cabri II tools													
Grades	**Group**	**Students**	**Categories of strategies leading to**									**Total Strat.**	**Path of strategies per student**
			Congruent triangles				**Equivalent (not exclusively congruent) triangles**						
			S1	**S2**	**S3**	**S4**	**G1**	**G2**	**G3**	**G4**	**G5**		
1st	**A1**	**P1**		5	1, 2, 4		3	6				6	S3 **G1** S3 S2 **G2**
		P2			2, 3	4	1	5 , 6				6	**G1** S3 S4 **G2**
		P3	2		1, 3, 4			6	5			6	S3 S1 S3 **G3 G2**
	A2	**P4**	1			2		3				3	S1 S4 **G2**
		P5	1		3			4		2		4	S1 **G4** S3 **G2**
	A3	**P6**			1, 3		2					3	S3 **G1** S3
		P7	1		2, 6, 7	3	4	5				7	S1 S3 S4 **G1** **G2** S3
		P8			1, 3		4	5		2		5	S3 **G4** S3 **G1 G2**
		Total str	**4**	**1**	**16**	**3**	**5**	**8**	**1**	**2**	**0**	**40**	
2nd	**B1**	**P9**			2, 3			6	4	1	5	6	**G4** S3 **G3 G5 G2**
		P10			1, 3, 4		5	7	2		6	7	S3 **G3** S3 **G1 G5 G2**
		P11			5, 6	2		3,4			1	6	**G5** S4 **G2 S3**
	B2	**P12**			2, 3	4,5				1		5	**G4** S3 S4
		P13			2	1	3					3	S4 S3 **G1**
		P14			1, 3, 4		2	5			6	6	S3 **G1** S3 **G2 G5**
	B3	**P15**		4	2, 3		1					4	**G1** S3 S2
		P16			1, 3, 4		2					4	S3 **G1** S3
		P17		2	5,6, 7	8, 9		3		1	4	9	**G4** S2 **G2 G5** S3 S4
		Total str	**0**	**2**	**21**	**6**	**5**	**6**	**2**	**3**	**5**	**50**	
3rd	**C1**	**P18**			1, 3		2					3	S3 **G1**
		P19			3			2		1		3	**G4 G2** S3
		P20	1		3, 4, 5		2					5	S1 **G1** S3
	C2	**P21**			2, 3, 4		1					4	**G1** S3
		P22			2, 3, 4		1					4	**G1** S3
		P23			1, 2, 3			4				4	S3 **G2**
	C3	**P24**			1, 6, 7	4	2		5		3	7	S3 **G1 G5** S4 **G3** S3
		P25	2		1		3					3	S3 S1 **G1**
		Total str	**2**	**0**	**19**	**1**	**6**	**2**	**1**	**1**	**1**	**33**	
All grades		**Total str.**	**6**	**3**	**56**	**10**	**17**	**16**	**4**	**6**	**5**	**123**	
Total students involved			**6**	**3**	**24**	**8**	**16**	**14**	**4**	**6**	**6**		

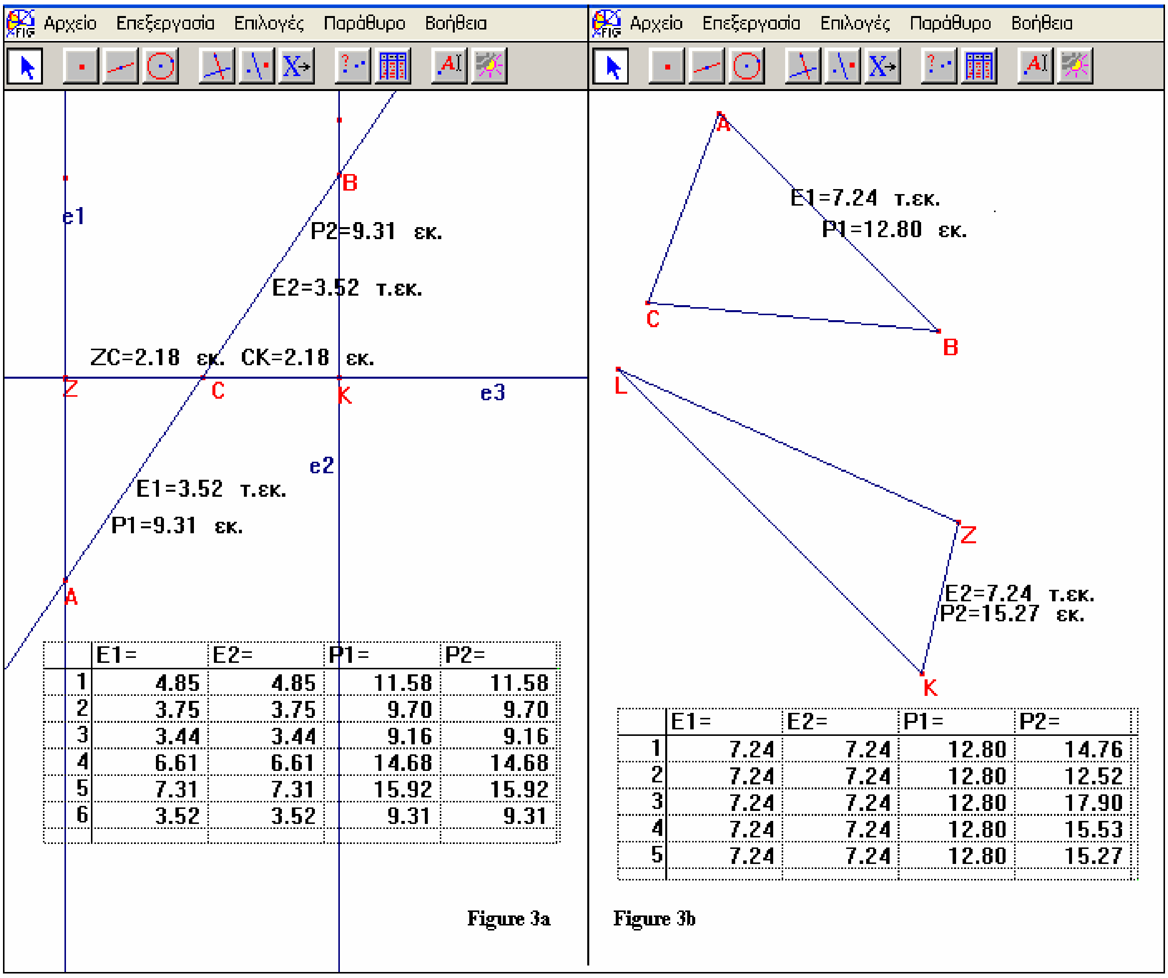
Table III

Categories of possible strategies leading to the construction of congruent triangles in the context of Cabri-Geometry II
Strategies constructed by:
S1: using the 'eye': Starting from the construction of a first triangle, construct a second triangle, and by using perception and possible control by measurement tools, try to obtain a triangle congruent to the first one.
S2: preserving lengths or lengths and angles of the original triangle.
S3: using Cabri-commands for geometrical transformations: • Translation ***(****Strategy S3a****)*** • Reflection about an axis *(Strategy S3b)* • Symmetry ***(****Strategy S3c)* • Rotation ***(****Strategy S3d)*
S4: Splitting polygons: eg. Splitting: • an isosceles triangle using a perpendicular bisector (*strategy S4a****)*** • a rectangle & a square into two equivalent triangles by using one of its diagonals (*strategy S4b)* • a regular polygon into a number of equivalent triangles by using all its diagonals (*strategy S4c)* • a parallelogram into two equivalent triangles by using one of its diagonals (*strategy S4d)*
Categories of possible strategies leading to the construction of equivalent not exclusively congruent triangles in the context of Cabri-Geometry II
Strategies constructed by:
G1: using the 'drag' mode in combination with automatic area measurement: • Constructing two triangles ABC and ZKL • Measuring the area of the triangles ABC and ZKL automatically • Dragging the vertices of the triangle ZKL to find different instances where its area is equal to that of the triangle ABC.
G2: conserving the length of the base and its distance from the opposite vertex in a triangle. • Constructing: two parallel lines e1 and e2, a segment BC on the line e1, a point A on the line e2 and the triangle ABC • Automatically measuring the area of the triangle ABC • Dragging point A on line e2 • Tabulating (or not) the area of the triangles constructed while point A is dragged
G3: splitting a triangle into two equivalent triangles using a median.
G4: measuring areas using area-units: • Constructing a triangle ABC • Illuminating the square grid provided by Cabri. • Measuring the area of this triangle by calculating the square units of the square grid • Constructing another triangle ZKL with an area consisting of the same number of square units as the triangle ABC
G5: using area formulae. Constructing an original triangle ABC and trying to construct another triangle equivalent to the original by conserving the product of the lengths of its base and its respective altitude.

Figure 1

Strategy S5
• Constructing two parallel lines e1 and e2 • Constructing a line e3 perpendicular to lines e1 and e2 which meets them at points Z and K respectively • Constructing the segment ZK • Constructing a line e4, which meets lines e1, e2 and e3 at points A, B and C respectively • Constructing the triangles ACZ and BCK and measuring their areas automatically • Dragging point C on segment ZK until the triangles above have equal areas • Rotating line e4 around point C and exploring the equivalence of area of the triangles ACZ and BCK *Tools used:* parallel and perpendicular lines, lines, segments, triangle, the 'drag mode' operation, rotation about an angle and around a point, automatic area measurement, automatic tabulation of numerical data.

Figure 2.



	E1=	E2=	P1=	P2=
1	4.85	4.85	11.58	11.58
2	3.75	3.75	9.70	9.70
3	3.44	3.44	9.16	9.16
4	6.61	6.61	14.68	14.68
5	7.31	7.31	15.92	15.92
6	3.52	3.52	9.31	9.31

	E1=	E2=	P1=	P2=
1	7.24	7.24	12.80	14.76
2	7.24	7.24	12.80	12.52
3	7.24	7.24	12.80	17.90
4	7.24	7.24	12.80	15.53
5	7.24	7.24	12.80	15.27

Figure 3

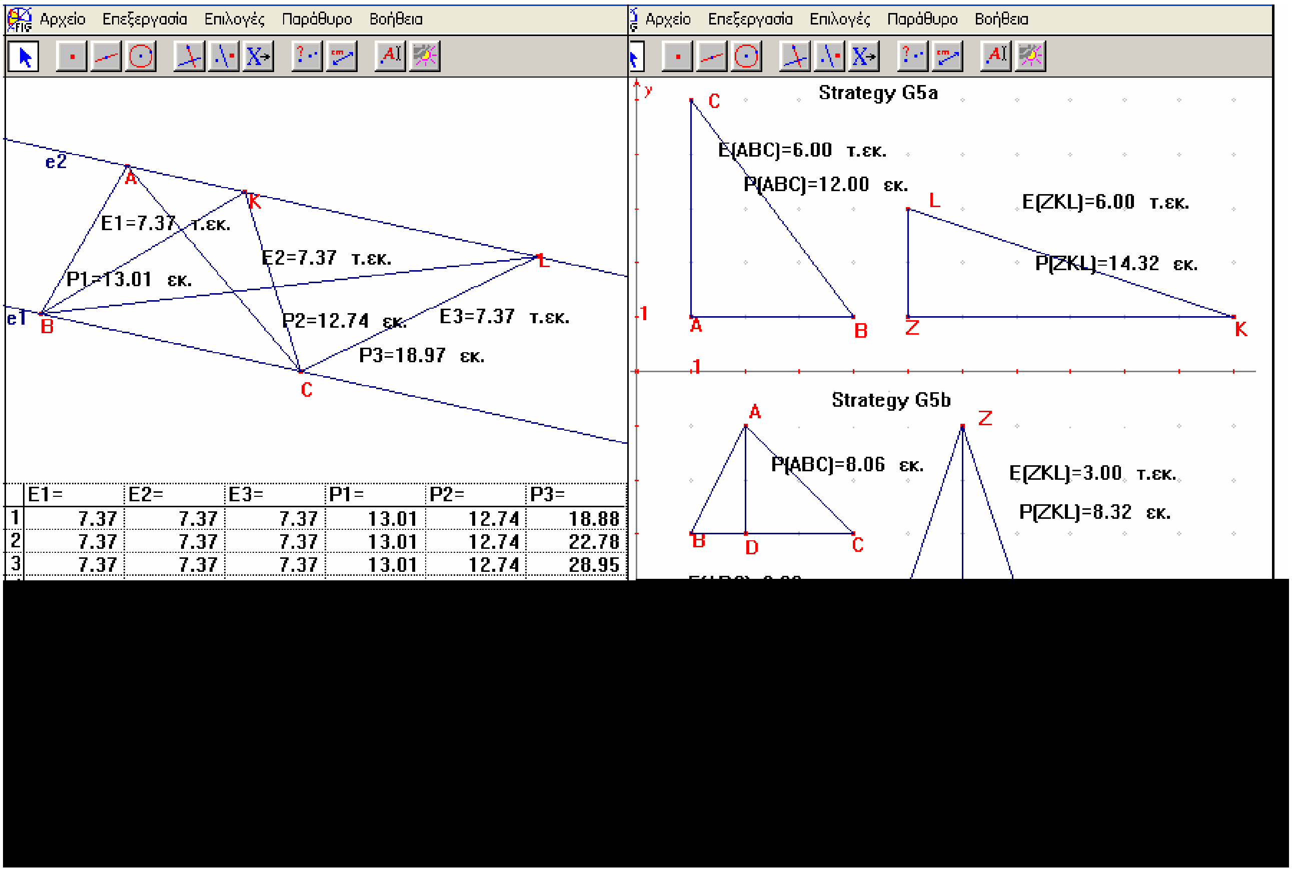

Αρχείο Επεξεργασία Επιλογές Παράθυρο Βοήθεια
e2
A
K
E1=7.37 τ.εκ.
E2=7.37 τ.εκ.
P1=13.01 εκ.
L
e1
B
P2=12.74 εκ.
E3=7.37 τ.εκ.
P3=18.97 εκ.
C
E1= E2= E3= P1= P2= P3=
1 7.37 7.37 7.37 13.01 12.74 18.88
2 7.37 7.37 7.37 13.01 12.74 22.78
3 7.37 7.37 7.37 13.01 12.74 28.95
Αρχείο Επεξεργασία Επιλογές Παράθυρο Βοήθεια
Strategy G5a
C
E(ABC)=6.00 τ.εκ.
P(ABC)=12.00 εκ.
L
E(ZKL)=6.00 τ.εκ.
P(ZKL)=14.32 εκ.
A
B
Z
K
Strategy G5b
A
Z
P(ABC)=8.06 εκ.
E(ZKL)=3.00 τ.εκ.
P(ZKL)=8.32 εκ.
B
D
C

Figure 4

Strategy G5a
• Constructing a right-angled triangle ABC ($A=90^o$) • Measuring the length of perpendicular sides AB and AC of this triangle by using the square grid provided by Cabri • Calculating the area of this triangle by using the area formulae: Area =(½)* [(AB)*(AC)], *=multiplication • Trying to find other pairs of numbers x, y and putting them in the position of AB and AC in the area formulae mentioned above to produce results equal to the calculated area of the original triangle • Constructing other right-angled triangles with perpendicular sides AB and AC which have lengths that are produced in the following way: AB = x*a, and AC =y*a, where a= the length of the side of the square unit of the used grid.
Strategy G5b
• Constructing a triangle ABC and its altitude AD • Measuring the length of the base BC and of the altitude AD • Calculating the area of this triangle by using the area formulae: Area =(½)* [(BC)*(AD)], *=multiplication • Constructing another triangle ZKL with base KL and altitude ZT equal to the length of the segments BC and AD respectively. The length of the segment BD was different from the length of the segment KT.
Strategy G5c
• Constructing a triangle ABC and its altitude AD • Measuring the length of the base BC and of the altitude AD • Calculating the area of this triangle by using the area formulae: Area =(½)* [(BC)*(AD)], *=multiplication • Constructing another triangle ZKL with base KL and altitude ZT equal to the length of the segments AD and BC respectively. • Repeating this construction by sliding the altitude ZT on base KL

Figure 5.

Strategy	Transforming an original triangle into other, equivalent ones by splitting polygons
	Constructing an original triangle ABC and automatically measuring its area and then..
S4b	• Duplicating the area of the original triangle • Constructing a rectangle & a square, using the 'regular polygon' tool with double the area of the original triangle • Splitting the constructed quadrilateral into two equivalent triangles by using one of its diagonals
S4c	• Multiplying the area of the original triangle by six • Constructing a regular hexagon with area equal to six times that of the original triangle • Splitting the constructed regular hexagon into six equivalent triangles by using all its diagonals
G3	• Duplicating the area of the original triangle • Constructing another triangle DEZ with double the area of the original triangle • Splitting the triangle DEZ using the median

Figure 6

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