CHAPTER 9

Balance Theory and Social Inequalities

We shall turn in the remainder of these notes to applications of signed and weighted graphs and digraphs. A graph or digraph is signed if there is a sign (+ or −) on each edge or arc. It is called weighted if there is a real number on each edge or arc. We shall sometimes consider signed graphs or digraphs as special cases of weighted graphs or digraphs, by replacing a sign + or − by a weight +1 or −1. In a signed graph or digraph, we shall associate a sign with a chain or path. The sign will be + if the number of − signs on the chain or path is even, and − otherwise.

In this chapter, we shall consider some problems of interest to social scientists. One main thrust of the work is the attempt to understand social inequalities. What is it about such characteristics as sex, race, occupation, education, and so on that leads to inequalities in social interaction? We shall begin with the theory of balance, which is a forerunner for much of the work on social interaction.

9.1. The theory of balance. A great deal of work in twentieth century sociology has concerned itself with the behavior of small groups of individuals. Perhaps the simplest approach to studying such a group is to draw a digraph in which the individuals are the vertices, and in which there is an arc from vertex x to vertex y if x is in some relation to y, for example, if x likes y, x associates with y, x chooses y for a business partner, etc. Such a digraph is sometimes called a sociogram. Many of the relationships of interest have natural opposites, for example likes/dislikes, associates with/avoids, and so on. In that case, we can include two different relations in one digraph by using two different kinds of arc, or by using signs to distinguish them. Then, the presence of an arc means that one of the relationships is present, and the + indicates one of the relationships, the positive one, while the − indicates the other relationship. For example, we might let an arc from x to y mean that x has strong feelings toward y, and put a + if these feelings are liking, a − if they are disliking. We obtain a signed digraph.

Let us for the sake of discussion deal with the concrete relation liking/disliking and let us assume that it is symmetric, so that we can summarize the information in a signed graph. The possible signed graphs if there are three individuals all of whom have strong feelings toward each other are shown in Fig. 9.1. Going back to the work of Heider (1946), it has been observed that groups of types I and III tend to work well together, work without tension, and so on. Groups of types II and IV do not. For example, in a group of type II, b likes both a and c, and would like to cooperate with them, but a and c dislike each other.
This causes tension. The same kind of tension does not appear in a group of type III, where $a$ and $b$ like each other, and both dislike $c$. They are perfectly content to let $c$ work on his own, and $c$ is quite satisfied with this arrangement.

Sociologists have used the imprecise term balance to describe groups which work well together, lack tension, etc. In general, groups of types I and III tend to be balanced, while groups of types II and IV do not. But what about groups whose signed graphs are more complicated? The signed graphs of types I and III can be distinguished from those of types II and IV because the former form circuits of positive sign (an even number of $-$ signs) while the latter form circuits of negative sign. This observation led Cartwright and Harary (1956) to suggest calling a signed graph and hence its underlying group of individuals balanced if every circuit was positive. In this sense, the signed graph of Fig. 9.2 is balanced, for the circuits are $a, b, d, a; b, c, d, b; d, e, f, e;$ and $a, b, c, d, a,$ and each of these has exactly two $-$ signs. The following theorem characterizes balanced signed graphs.

**Theorem 9.1 (Harary (1954)).** A signed graph is balanced if and only if the vertices can be partitioned into two classes so that every edge joining vertices within a class is $+$ and every edge joining vertices between classes is $-$.

To illustrate this theorem, we observe that two classes which provide such a partition for the signed graph of Fig. 9.2 are $\{a, c, d, f\}$ and $\{b, e\}$. It is easy to see why the existence of such a partition implies balance. For every circuit must begin in one of the classes and end there, and so has only an even number of crossings between classes, and hence only an even number of $-$ signs. The converse is also easy to prove. One shows without loss of generality that the signed graph is connected. Then, choosing a vertex $u$ at random, one defines one class to be all the vertices joined to $u$ by a positive chain, and the other class to be all remaining vertices. The balance hypothesis is exactly what is needed to prove that this partition has the desired properties.

Theorem 9.1 can be considered a generalization of König's theorem (Theorem 6.1), which says that a graph is bipartite if and only if it has no odd circuits. We may obtain König's theorem from Harary's by taking all edges to be $-$. The theory of balance suggests that groups with balanced signed graphs will exhibit lack of tension, work well together, and so on. For a discussion of tests of this theory, the reader is referred to Taylor (1970).

Balance theory has been applied to a variety of problems outside of sociology. In particular, it has been applied to the study of international relations, where...
the vertices become nations and the relation is allied with/allied against. It has been applied to political science, where the vertices are politicians and the relation is agrees with/disagrees with. It has been applied to the analysis of literature, where the vertices are characters in a novel, play, or short story, and the relation is liking/disliking. It is hypothesized that at a stage of tension in such a piece of work, the main characters will exhibit an unbalanced signed graph. Later, the tension will be resolved by changing a sign of a major relationship to obtain balance.

For a more detailed discussion of balance theory, with many references, see Taylor (1970). See also Roberts (1976a, § 3.1).

9.2. Balance in signed digraphs. Let us briefly discuss how the notion of balance generalizes to a signed digraph. It is natural to try to call a signed digraph balanced if every cycle is positive. However, this makes the signed digraph of Fig. 9.3 vacuously balanced, while there is tension in this situation: $a$ likes both $b$ and $c$, and would like to work with them both, but $c$ dislikes $b$.

An appropriate generalization of the balance concept is obtained by disregarding direction of arcs. To be precise, we say that a \textit{semipath} in a (signed) digraph is a sequence $u_1, a_1, u_2, a_2, \cdots, u_n, a_n, u_{n+1}$, where the $u_i$ are vertices, the $a_i$ are arcs, and $a_i$ is the arc $(u_i, u_{i+1})$ or the arc $(u_{i+1}, u_i)$. That is, in a semipath, arcs may be traversed in either direction. The \textit{length} of the semipath is $t$. In Fig. 9.4, $a, (a, c), c, (d, c), d$ is a semipath of length 2. A \textit{semicycle} is a semipath in which $u_1 = u_{t+1}$ and all the vertices $u_1, u_2, \cdots, u_t$ are distinct and all the arcs $a_1, a_2, \cdots, a_t$ are distinct. Thus, for example, in Fig. 9.4, $a, (a, b), b, (c, b), c, (a, c), a$ is a semicycle. So is $a, (b, a), b, (c, b), c, (a, c), a$. These are different as the first arc used differs in the two cases. The \textit{sign} of a semipath is defined analogously to the sign of a path. We say a signed digraph is \textit{balanced} if and only

Fig. 9.2. A balanced signed graph.

Fig. 9.3. An unbalanced signed digraph with no negative cycles.
if every semicycle is positive, i.e., has an even number of − signs. This definition seems to be an appropriate generalization of the definition in the symmetric case.

9.3. Degree of balance. It is a little simplistic to say that every group is either completely balanced or completely unbalanced. Rather, it probably makes more sense to speak of degrees of balance.

Let us discuss briefly some ways of measuring balance in a signed graph or digraph. One natural way to measure balance is to use the ratio \( p/t \) of the number of positive circuits (semicycles) to the total number of circuits (semicycles). An alternative is to use the ratio \( p/n \), where \( n \) is the number of negative circuits (semicycles). Variants on this approach take account of the length of a circuit (semicycle), counting shorter circuits (semicycles) as being at least as important, and possibly more important, than longer ones. One way of taking account of length is the following. Let \( p_m \) be the number of positive circuits (semicycles) of length \( m \), \( n_m \) the number of negative circuits (semicycles) of length \( m \), and \( t_m = p_m + n_m \). Then if \( f(m) \) is a measure of the relative importance of circuits (semicycles) of length \( m \), we might use

\[
\frac{\sum_m p_m f(m)}{\sum_m t_m f(m)} \tag{9.1}
\]

or

\[
\frac{\sum_m p_m f(m)}{\sum_m n_m f(m)} \tag{9.2}
\]

The function \( f(m) \) might be monotone nondecreasing, and might be something like \( f(m) = 1/m \), \( f(m) = 1/m^2 \), \( f(m) = 1/2^m \), etc. The function \( f(m) = 1 \) gives the measures \( p/t \) and \( p/n \). For a discussion of the measures \( p/t \) and \( p/n \), see Harary (1959c). For a discussion of the measures (9.1) and (9.2), and an axiomatic derivation of them, see Norman and Roberts (1972a,b).
An alternative is to count the smallest number of signs whose negation would result in balance. This count is called the line index for balance. Harary (1959c) proves that in a signed graph (digraph) the line index is the same as the smallest number of edges (arcs) whose removal results in balance.

The measures \( (9.1) \) and \( (9.2) \) are equivalent in the sense that once \( f(m) \) has been determined, one signed graph (digraph) is more balanced than another under \( (9.1) \) if and only if this is also the case under \( (9.2) \). However, the measure \( (9.1) \) and the line index for balance are not equivalent, i.e., there are simple examples of signed graphs (digraphs) \( G_1 \) and \( G_2 \) so that \( G_1 \) is more balanced under the line index than \( G_2 \) but less balanced under measure \( (9.1) \). For a discussion of other measures of balance, see Taylor (1970).

9.4. Distributive justice. So far, we have dealt with the situation where the vertices of a signed graph or digraph are individuals in a group. Interesting results are obtained if we allow the vertices to be other variables as well. In this section we shall use this idea to discuss the theory of distributive justice in sociology. This theory is concerned with the relation between such characteristics as sex, race, hair color, etc., and such goal objects or rewards as salaries, promotions, privileges, etc. The key idea is one of expectation: an individual compares his characteristics and rewards to those of others, and wants to see if justice has been done.

A theory of distributive justice is sketched out in Zelditch, Berger, and Cohen (1966) and Berger, Zelditch, Anderson and Cohen (1972). To treat the simplest case of this theory graph-theoretically, we follow Norman and Roberts (1972b). Let \( P \) and \( O \) be two individuals, and let \( P' \) be \( P \) considered as a referrent for evaluation by \( P \). (\( P' \) can be thought of as the image \( P \) has of himself.) We study the situation from the point of view of \( P \). We study one characteristic and one goal object, assume that the characteristic is relevant to the goal object, and that the characteristic and goal object can each have one of two states, high or low. Let \( \text{GO}(P) \) and \( \text{GO}(O) \) denote the states of the goal objects possessed by \( P \) and by \( O \) respectively. We build a signed digraph as follows. The vertices are \( P, P', O, \text{GO}(P), \) and \( \text{GO}(O) \). We draw an arc from \( P \) to \( \text{GO}(P) \) and to \( \text{GO}(O) \). On each of these arcs, we put a sign indicating the evaluation by \( P \) of the state, \(+\) for the high state, \(-\) for the low state. We draw arcs from \( P \) to \( P' \) and \( P \) to \( O \). On each of these arcs, we put a sign indicating the evaluation by \( P \), \(+\) if the individual in question (\( P' \) or \( O \)) possesses the high state of the characteristic, \(-\) if he possesses the low state of the characteristic. Finally, we draw arcs from \( P' \) to \( \text{GO}(P) \) and \( O \) to \( \text{GO}(O) \) with \(+\) signs indicating possession. The signed digraph obtained is shown in Fig. 9.5. For example, the characteristic might be education and the goal object salary. We feel education is relevant to salary. An individual is positively evaluated if he has a high level of education. A salary is positively evaluated if it is high.

Under the assumption that the parameter \( f(m) \) is positive, we have calculated the balance measure \( (9.1) \) for each choice of sign in this digraph. The results are shown in Table 9.1 below. Cases 1, 4, 13, and 16 correspond to what Berger et
al. (1972) call Justice. In each case, the state of the characteristic matches the state of the reward. In Cases 6, 7, 10, and 11, both individuals are unjustly rewarded. In Case 10, for example, $P$ is over-rewarded (he has a low level of education and a high salary) while $O$ is under-rewarded. This is a situation of Guilt. The remaining cases have one individual justly rewarded, but the other not. The balance measure indicates that the situations of Justice are perfectly balanced, the situations where both individuals are unjustly rewarded are perfectly unbalanced, and the situations where one individual is unjustly rewarded are in between. By adding more information to this signed digraph, Norman and Roberts (1972b) show how further distinctions among the different cases can be made.

For example, we can study coalition formation to distinguish the four cases where both individuals are unjustly rewarded. We draw an arc from $P'$ to $O$ representing attraction/repulsion and ask whether the arc should receive a + or − sign, + for attraction, − for repulsion. We obtain the signed digraph of Fig. 9.6. We have calculated the numerator of the balance measure (9.1) for the two choices of sign on this arc in each of the cases 6, 7, 10, and 11. (The denominator is always the same in all cases, since all have the same semicycles.) The results are shown in Table 9.2. Suppose we assume that $P'$ and $O$ form a coalition if and only if the signed digraph is more balanced with attraction than with repulsion. Now coalition formation occurs in Cases 6 and 11 if and only if

$$f(3) + f(5) > 2f(4)$$

and in Cases 7 and 10 if and only if

$$2f(4) > f(3) + f(5).$$

Hence, we conclude that either there is no coalition formation ($f(3) + f(5) = 2f(4)$), or it takes place only in situations where both individuals are treated with equal injustice (Cases 6 and 11), or it takes place only where the two individuals are treated with opposite injustice (Cases 7 and 10). Notice that this conclusion follows independently of the specific value of the parameter $f$. Adding more

![Fig. 9.5. Signed digraph for the distributive justice situation.](image-url)
information to the digraph allows the conclusion that coalition formation will take place in situations of equal injustice. We shall not go into further detail here. Rather, we simply mentioned this example to illustrate the kind of analysis which is sometimes made using signed digraphs where the vertices are other than just individuals and where the arcs represent different kinds of relationships.

**Table 9.1**

Relative balance in 16 cases of signed digraph of Fig. 9.5.

<table>
<thead>
<tr>
<th>Case Number</th>
<th>PP</th>
<th>PG(P)</th>
<th>PO</th>
<th>PG(O)</th>
<th>Balance</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>1</td>
<td>Justice</td>
</tr>
<tr>
<td>2</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>1/2</td>
<td>Other unjustly rewarded (under-reward)</td>
</tr>
<tr>
<td>3</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>1/2</td>
<td>Other unjustly rewarded (over-reward)</td>
</tr>
<tr>
<td>4</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>1</td>
<td>Justice</td>
</tr>
<tr>
<td>5</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>1/2</td>
<td>Self unjustly rewarded (under-reward)</td>
</tr>
<tr>
<td>6</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>0</td>
<td>Both unjustly rewarded</td>
</tr>
<tr>
<td>7</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>0</td>
<td>Both unjustly rewarded</td>
</tr>
<tr>
<td>8</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1/2</td>
<td>Self unjustly rewarded (under-reward)</td>
</tr>
<tr>
<td>9</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>1/2</td>
<td>Self unjustly rewarded (over-reward)</td>
</tr>
<tr>
<td>10</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>0</td>
<td>Both unjustly rewarded</td>
</tr>
<tr>
<td>11</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>0</td>
<td>Both unjustly rewarded</td>
</tr>
<tr>
<td>12</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>1/2</td>
<td>Self unjustly rewarded (over-reward)</td>
</tr>
<tr>
<td>13</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>1</td>
<td>Justice</td>
</tr>
<tr>
<td>14</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>1/2</td>
<td>Other unjustly rewarded (under-reward)</td>
</tr>
<tr>
<td>15</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>1/2</td>
<td>Other unjustly rewarded (over-reward)</td>
</tr>
<tr>
<td>16</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1</td>
<td>Justice</td>
</tr>
</tbody>
</table>
9.5. Status organizing processes and social inequalities. Since the early twentieth century, sociologists have been concerned with status-organizing processes, processes by which differences in evaluations and expectations of individuals affect social interactions. A major goal of studying such processes is the explanation of social inequalities. The theory of distributive justice was an early theory in the work of Berger and his colleagues, which tried to explain the tension resulting from violation of expectations by showing that certain signed digraphs were unbalanced, or relatively unbalanced. In Berger et al. (1977), chains in a signed graph are used to study induced expectations. The basic idea is that the variables are individuals, goal objects, states, and so on, much as in our discussion of § 9.4, except that direction of relationships is disregarded. In the resulting signed graphs, if there is a chain from an individual $P$ to a goal object $G$, the sign of this chain indicates whether the individual is expected to be rewarded with the high or the low state of the goal object. If there are several chains from $P$ to $G$ of the same sign, this leads to stronger expectations, while if there are chains of opposite sign, this leads to inconsistent expectations (there is imbalance). The expectations are calculated quantitatively using a parameter much like $f(m)$, and are used to predict reactions of one individual to another when they disagree.

9.6. Strengths of likes and dislikes. One of the weaknesses of the signed graph approach to balance is that it omits strengths of likes and dislikes. One approach which attempts to use such strengths is due to Hubbell, Johnson, and Marcus (1978). Let the individuals in a group be labeled as $1, 2, \cdots, n$. Let $s^{(t)}_{ij}$ be the sentiment of $i$ for $j$ at time $t$. How do sentiments change over time? Hubbell et al. assume changes take place only at discrete times $t = 0, 1, 2, \cdots$ and make the following assumption. The sentiment $s_{ik}$ of $i$ for $k$ can be changed by sentiments of $i$ for $j$ and $j$ for $k$—this is called an inductive change—and by sentiments of $i$ for $j$ and $k$ for $j$—this is called a comparative change.
change. Assuming these effects take place equally, Hubbell et al. specify that

\[(9.3) \quad s_{ik}^{(t+1)} = \frac{1}{2} \left( \sum_j s_{ij} s_{jk} + \sum_j s_{ij} s_{kj} \right).\]

In terms of matrices, if \( S_t = (s_{ij}^{(t)}) \), then (9.3) becomes

\[(9.4) \quad S_{t+1} = \frac{1}{2} (S_t^2 + S_t S_t^T)\]

where \( S_t^T \) is the transpose of \( S_t \).

Given a matrix \( S \), we can associate a sign pattern with it. For example, the matrix

\[(9.5) \quad S = \begin{pmatrix} 75 & -6 & 0 \\ -1 & 7 & 0 \\ 0 & 0 & 0 \end{pmatrix}\]

has the sign pattern

\[(9.6) \quad \begin{pmatrix} + & - & 0 \\ - & + & 0 \\ 0 & 0 & 0 \end{pmatrix}.\]

The sign pattern defines a signed digraph in a natural way. Under the assumption (9.4), the sequence of matrices might or might not approach a limit. (Hubbell et al. discuss this problem matrix-theoretically.) Of more interest to us is the corresponding sequence of sign patterns. Does this approach a limit, i.e., is there some point beyond which all of the sign patterns are the same? If so, does this sign pattern correspond to a balanced signed digraph? How can we tell if a given matrix \( S = S_1 \) gives rise to a sequence of sign patterns which eventually stabilizes? How can we tell if the sequence eventually stabilizes to a pattern corresponding to a balanced signed digraph? Does the answer depend on the specific numbers in \( S \)?

We have asked lots of questions. Let us give an example. The matrix \( S \) of (9.5) has the property that if \( S = S_1 \), then all matrices \( S_t \) defined by (9.4) have the same sign pattern (9.6). The corresponding signed digraph is balanced. Thus, we know that if the initial sentiments of the individuals are given by (9.5), and if they change according to the Hubbell et al. model (9.4), then the group will always be balanced. The conclusion depends only on the sign pattern of \( S \), and not on the specific entries. The advantage of the model we have discussed over those using only signs is that in the latter, changes of sign can take place only abruptly, while here sentiments can change gradually.

The questions we have raised (which are also raised by Hubbell et al.) seem to be difficult ones, and to our knowledge, no one has made progress on them.
However, we shall see in the next two chapters that one can answer a large number of similar questions which arise in public policy situations, economic analysis, applied ecology, and the like. These questions deal with matrices whose elements are known only up to sign. The questions then ask: can we draw various conclusions about such things as systems of linear equations or systems of differential equations which use these matrices for coefficients, if we know the entries in the matrices only up to sign? We shall see that we can draw many such conclusions, by making use of the signed digraph corresponding to the sign pattern.